

# Manifestly gauge invariant theory of the nonlinear cosmological perturbations in the leading order of the gradient expansion

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## Abstract

In the full nonlinear cosmological perturbation theory in the leading order of the gradient expansion, all the types of the gauge invariant perturbation variables are defined. The metric junction conditions across the spacelike transition hypersurface are formulated in a manifestly gauge invariant manner. It is manifestly shown that all the physical laws such as the evolution equations, the constraint equations, and the junction conditions can be written using the gauge invariant variables which we defined only. Based on the existence of the universal adiabatic growing mode in the nonlinear perturbation theory and the  $\rho$  philosophy where the physical evolution are described using the energy density  $\rho$  as the evolution parameter, we give the definitions of the adiabatic perturbation variable and the entropic perturbation variables in the full nonlinear perturbation theory. In order to give the analytic order estimate of the nonlinear parameter  $f_{NL}$ , we present the exponent evaluation method. As the models where  $f_{NL}$  changes continuously and becomes large, using the  $\rho$  philosophy, we investigate the non-Gaussianity induced by the entropic perturbation of the component which does not govern the cosmic energy density, and we show that in order to obtain the significant non-Gaussianity it is necessary that the scalar field which supports the entropic perturbation is extremely small compared with the scalar field which supports the adiabatic perturbation.

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## §1 Introduction and summary

In the inflationary scenario, the quantum fluctuations of the scalar fields driving the inflationary expansion of the universe are the origins, that is the seed perturbations of the temperature fluctuations of the cosmic microwave background radiation (CMB) and the cosmic large scale structures such as galaxies and clusters of galaxies. These seed perturbations generated in the horizon during the inflationary expansion are stretched and go out of the horizon. They stay outside the horizon until they return into the horizon in the Friedman expansion stage. Therefore in order to compare the theory with the observation, it is necessary to solve the evolutions of the cosmological perturbations on superhorizon scales using the concrete theoretical models such as the various inflation and the reheating scenarios. [3] [4] [5] [2] [6] [7] [8] [9] Fortunately as for the evolutions of the cosmological perturbations in the long wavelength limit, the exact solution is constructed in terms of the evolution of the corresponding locally homogeneous universe; as for the linear perturbations in the papers [10] [5] [11] [8] and as for the full nonlinear perturbations in the papers [13] [14] [9]. The final form of the exact solutions of the evolutions of the cosmological perturbations in the long wavelength limit is established by the Kodama Hamazaki construction (KH construction), as for the linear perturbations in the papers [5] [8], and as for the full nonlinear perturbations in the paper [9]. In the KH construction, the physical quantities related with the exactly homogeneous universe, such as the scalar quantity perturbations, are given as the solutions of the evolution equations of the corresponding locally homogeneous universe and the physical quantities not related with the exactly homogeneous universe, such as the vector quantity perturbations, are given by solving the first order evolution equations, that is, the spatial components of the Einstein equations. It was shown that the second order evolution equations of the spatial unimodular metric including the information of the adiabatic decaying mode is exactly solvable. In the present paper, we use the KH construction.

The general theory of relativity is a gauge theory. When we solve the equations of the general theory of relativity, the nondynamical gauge modes are contained in the solutions. Therefore in order to extract the dynamical modes only, it is desirable to write down the equations in terms of the gauge invariant variables only. In the linear perturbation theory, the program of the gauge invariant perturbation variables was first performed in the paper [1], and was extended so that we can treat the multicomponent systems [17] [18] [19]. In the second order perturbation theory, the gauge invariant perturbation theory was constructed in the papers [36] [37] [15]. In our previous paper [9], the full nonlinear perturbation theory in the leading order of the gradient expansion was constructed and several main definitions of the gauge invariant perturbation variables including the nonlinear Bardeen parameters [1] [19] [12] [14] [9] were presented. In the present paper, in a more general way, definitions of all the types of the gauge invariant perturbation variables are constructed and it is manifestly shown that all the perturbation equations of the physical laws such as the evolution equations, the constraint equations and the metric junction conditions can be written by using the gauge invariant perturbation variables which we defined only. By solving the equations of the gauge invariant formulation of the full nonlinear perturbations in the leading order of the gradient expansion which we formulated, we can extract the full nonlinear physically meaningful, dynamical information of the cosmological perturbations on superhorizon scales.

In order to interpret the physics of the evolutionary behaviors of the cosmological perturbations, the Adiabatic/Entropic decomposition (A/E decomposition) of the cosmological perturbations is efficient. The essence of the A/E decomposition is in defining the adiabatic perturbation variable and the entropic perturbation variables. Although the linear version of the A/E decomposition has been already established [17] [18], the satisfactory definitions of the adiabatic perturbation variable and the entropic perturbation variables in the nonlinear perturbation theory has not been completed yet. In the present paper, we give the definitions of the adiabatic perturbation variable and the entropic perturbation variables which can be used in the nonlinear perturbation theory by using the fact that the universal adiabatic growing mode always exists in the solutions in the nonlinear perturbation theory in the long wavelength limit. [5] That is, we call the perturbation variable which does not vanish for the universal adiabatic growing mode the adiabatic perturbation variable and we call the perturbation variable which vanishes for the universal adiabatic growing mode the entropic perturbation variable. In particular, the adiabatic/entropic perturbation variables which are defined under the  $\rho$  philosophy where the evolutions of the system are traced by choosing the energy density  $\rho$  as the evolution parameter, have desirable properties. All the perturbation variables in this set are continuous across the metric junction hypersurface which is defined by  $\rho = \text{const}$  such as the slow rolling-oscillatory transitions of the scalar fields and the reheating transitions. The evolution equations of the perturbation variables in this set which can be derived by quite easy calculation have very simple expression. The adiabatic perturbation variable in this set is the well-known Bardeen parameter. [1] [19] [12] [14] [9]

In the near future, more precise observations of CMB will be performed and it is expected that the information of the nonlinearity of the CMB fluctuations will be obtained. Motivated by the observational advancement, the models which generate the significant nonlinearity characterized by the large non-Gaussianity parameter  $f_{NL}$  [35] have been proposed; the inhomogeneous end of the inflation [28] [29], the modulated reheating [31], the curvaton scenario [32], the vacuum dominated inflation [33] [34]. The former two cases are related with the metric junction hypersurface which cannot be defined by  $\rho = \text{const}$  and the large non-Gaussianities  $f_{NL}$  are generated discontinuously on the transition hypersurface. In the latter two cases, the non-Gaussianity  $f_{NL}$  grows continuously and becomes very large transiently. In the present paper, we present the exponent evaluation method which enables us to give the analytic order estimates of the non-Gaussianities  $f_{NL}$  in these models. We discuss that the mechanisms which generate the large non-Gaussianities  $f_{NL}$  in the latter two cases are common, although in the first case in the latter two cases the cosmological term does not exist while in the second case in the latter two cases the cosmological term exists. In the latter two cases, the entropic perturbation of the component which does not govern the cosmic energy density can trigger the growth of the Bardeen parameter  $\zeta_n(\rho)$  and the non-Gaussianity  $f_{NL}$ , when the scalar fields which support the entropic perturbation are very small, since the influences of these small scalar fields on the Bardeen parameter  $\zeta_n(\rho)$  can become large.

The rest of the present paper is organized as follows. In the section 2, we give the definitions of all the types of the gauge invariant perturbation variables and show manifestly that in the long wavelength limit all the perturbation equations of all the physical laws derived by the general theory of relativity can be written in the gauge invariant manner.

In the section 3, under the  $\rho$  philosophy, we complete the A/E decomposition of the full nonlinear perturbations by giving the definitions of the adiabatic perturbation variable and the entropic perturbation variables. In the section 4, as the application of the A/E decomposition based on the  $\rho$  philosophy formulated in the previous section, we investigate the evolutions of the cosmological perturbations in the universe where the growth of the adiabatic perturbation variable called the Bardeen parameter [1] [19] [12] [14] [9] is induced by the entropic perturbation of the subdominant component. We evaluate the non-Gaussianity parameter  $f_{NL}$  by the exponent evaluation method. We present the condition for which the non-Gaussianity  $f_{NL}$  becomes large in the models where  $f_{NL}$  changes continuously.

## §2 the manifestly gauge invariant formulation of the nonlinear cosmological perturbation theory in the leading order of the gradient expansion

### 2.1 the evolution equations and the constraint equations

We consider the Einstein equations  $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$  where  $\kappa^2$  is expressed in terms of the Newtonian gravitational constant  $G$  as  $\kappa^2 = 8\pi G$ , using the  $3+1$  decomposition. [38] [16] [9] The Greek indices  $\mu, \nu, \dots$  run from 0 to 3 and the Latin indices  $i, j, \dots$  run from 1 to 3. The metric tensor  $g_{\mu\nu}$  is expressed as

$$g_{00} = -\alpha^2 + \beta_k \beta^k, \quad (2.1)$$

$$g_{0i} = \beta_i, \quad (2.2)$$

$$g_{ij} = \gamma_{ij}, \quad (2.3)$$

where  $\alpha$  is the lapse and  $\beta_i$  is the shift vector. The index of  $\beta_i$  is raised by  $\gamma^{ij}$  which is the inverse matrix of  $\gamma_{ij}$ . The spatial metric  $\gamma_{ij}$  is factorized as

$$\gamma_{ij} = a^2 \tilde{\gamma}_{ij}, \quad (2.4)$$

where  $\tilde{\gamma}_{ij}$  is the unimodular matrix whose inverse matrix is expressed as  $\tilde{\gamma}^{ij}$  and  $a$  is the scale factor. The energy momentum tensor of the total system  $T_{\mu\nu}$  is expressed as

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \quad (2.5)$$

where  $\rho$ ,  $P$  and  $u_\mu$  are the energy density, the pressure and the four velocity vector of the total system, respectively. Because of the normalization condition  $u^\mu u_\mu = -1$ ,  $u_\mu$  can be parametrized as

$$u_0 = -u^0 \{\alpha^2 - \beta^k (\beta_k + v_k)\}, \quad (2.6)$$

$$u_k = u^0 (\beta_k + v_k), \quad (2.7)$$

where  $u^0 = g^{0\mu} u_\mu$  is given by

$$u^0 = \{\alpha^2 - (\beta_k + v_k)(\beta^k + v^k)\}^{-1/2}, \quad (2.8)$$

$v_k$  is the three velocity of the total system and the index of  $v_k$  is raised by  $\gamma^{ij}$ .  $T_{\mu\nu}$  is expressed as

$$T_{\mu\nu} = \sum_{\alpha} T_{\alpha\mu\nu} + T_{S\mu\nu}, \quad (2.9)$$

where  $T_{\alpha\mu\nu}$  is the energy momentum tensor of the perfect fluid component  $\alpha$  and  $T_{S\mu\nu}$  is the energy momentum tensor of all the scalar fields.  $T_{\alpha\mu\nu}$  is expressed by (2.5) (2.6) (2.7) (2.8) where  $\rho$ ,  $P$ ,  $u_{\mu}$  and  $v_i$  are replaced with  $\rho_{\alpha}$ ,  $P_{\alpha}$ ,  $u_{\alpha\mu}$  and  $v_{\alpha i}$ , respectively.  $T_{S\mu\nu}$  is expressed by

$$T_{S\mu\nu} = \sum_a \partial_{\mu}\phi_a \partial_{\nu}\phi_a - \frac{1}{2} \left\{ \sum_a g^{\rho\sigma} \partial_{\rho}\phi_a \partial_{\sigma}\phi_a + 2U \right\} g_{\mu\nu}. \quad (2.10)$$

As for the scalar fields, since we cannot decide to which component  $a$  each term of the potential  $U$  belongs,  $T_{S\mu\nu}$  cannot be decomposed into  $T_{a\mu\nu}$ . In this way, the indices of the component  $A$  are divided into the perfect fluid indices  $\alpha$  and the scalar field indices  $a$ . The energy momentum transfer vectors of the perfect fluid component  $\alpha$  and the scalar field component  $a$  are expressed by

$$Q_{\alpha\mu} = Q_{\alpha} u_{\mu} + f_{\alpha\mu}, \quad u^{\mu} f_{\alpha\mu} = 0, \quad (2.11)$$

$$Q_{a\mu} = S_a \partial_{\mu}\phi_a, \quad (2.12)$$

where  $Q_{\alpha}$  and  $f_{\alpha\mu}$  are the energy transfer and the momentum transfer of the perfect fluid component  $\alpha$ , respectively and  $S_a$  is the source function of the scalar field component  $a$ . The energy momentum conservation gives

$$\sum_{\alpha} Q_{\alpha\mu} + \sum_a Q_{a\mu} = 0. \quad (2.13)$$

As for the perfect fluid component  $\alpha$ ,  $\nabla_{\mu} T_{\alpha\nu}^{\mu} = Q_{\alpha\nu}$  gives the equation of motion of the perfect fluid component  $\alpha$ . As for the scalar fields,  $\nabla_{\mu} T_{S\nu}^{\mu} - \sum_a Q_{a\nu} = 0$  can be expressed as the linear combination of  $\partial_{\nu}\phi_a$ . By assuming that the each coefficient of  $\partial_{\nu}\phi_a$  is separately vanishing, we can derive the phenomenological equation of motion of the scalar field  $\phi_a$ ,  $\square\phi_a - \partial U/\partial\phi_a = S_a$ .

Since we want to treat the cosmological perturbations on superhorizon scales, we put the gradient expansion assumptions by using the small parameter  $\epsilon$  characterizing the inverse of the long wavelength of the cosmological perturbations. Since the spatial scale of the inhomogeneity of all the physical quantities is of the order of  $1/\epsilon$ , we assign  $\partial_i = O(\epsilon)$ . As for the metric, we assign  $g_{0i} = O(\epsilon)$ . For arbitrary vector fields  $V_{\mu}$  satisfying  $V^{\mu}V_{\mu} = O(1)$  including  $u_{\mu}$ ,  $u_{\alpha\mu}$ , we assume that  $V_i = O(\epsilon)$ . Therefore  $\beta_i$ ,  $\beta^i$ ,  $v_i$ ,  $v^i$ ,  $v_{\alpha i}$ ,  $v_{\alpha}^i$  and  $f_{\alpha i}$  are of the order of  $\epsilon$ . As for the velocity vector of the total system and the perfect fluid component  $\alpha$ , the leading order of the gradient expansion can be expressed by

$$u_0 = -\alpha + O(\epsilon^2), \quad (2.14)$$

$$u_i = \frac{1}{\alpha}(v_i + \beta_i) + O(\epsilon^3), \quad (2.15)$$

$$u_{\alpha 0} = -\alpha + O(\epsilon^2), \quad (2.16)$$

$$u_{\alpha i} = \frac{1}{\alpha}(v_{\alpha i} + \beta_i) + O(\epsilon^3). \quad (2.17)$$

As for the momentum transfer vector of the perfect fluid component  $\alpha$ ,  $u^\mu f_{\alpha\mu} = 0$  gives

$$f_{\alpha 0} = 0 + O(\epsilon^2). \quad (2.18)$$

We consider the gauge transformation laws of all the physical quantities. The gauge transformation laws are written in terms of the Lie derivative. The Lie derivatives of the quantity with upper index and the quantity with lower index are expressed by

$$L(T)X^\mu = T^\rho \partial_\rho X^\mu - \partial_\rho T^\mu X^\rho, \quad (2.19)$$

$$L(T)X_\mu = T^\rho \partial_\rho X_\mu + \partial_\mu T^\rho X_\rho. \quad (2.20)$$

The Lie derivative of the tensor field of an arbitrary rank is given by the above two definitions and the Leibniz rule. Because of the gradient expansion assumption, the infinitesimal coordinate transformation generating the Lie derivative  $T^\rho \partial_\rho$  satisfies  $T^i = O(\epsilon)$ . Under the gradient expansion assumption, the Lie derivative of the scalar  $S$  is given by

$$L(T)S = T^0 \dot{S} + O(\epsilon^2), \quad (2.21)$$

and the Lie derivative of the vector  $V_\mu$  is given by

$$L(T)V_0 = T^0 \dot{V}_0 + \dot{T}^0 V_0 + O(\epsilon^2), \quad (2.22)$$

$$L(T)V_i = T^0 \dot{V}_i + \partial_i T^0 V_0 + O(\epsilon^3). \quad (2.23)$$

Under the gradient expansion scheme, it is possible that the quantity which is not a scalar, for example  $\gamma_{ij}$ , has the Lie derivative of the scalar type (2.21). So we expand the definitions of the scalar field and the vector field as follows.

**Definition** The physical quantity which has the Lie derivative (2.21) is called the scalar like object. The physical quantity which has the Lie derivative (2.22) (2.23) is called the vector like object.

Following these definitions, the physical quantities such as  $a$ ,  $\gamma_{ij}$ ,  $\tilde{\gamma}_{ij}$ ,  $\gamma^{ij}$ ,  $\tilde{\gamma}^{ij}$  are the scalar like objects and the  $\partial_\mu$  derivative of these quantities is the vector like object. We can demonstrate the following propositions easily.

**Proposition 1** For a scalar like object  $S$ ,  $\partial_\mu S$  is a vector like object.

Please use  $L(T)\partial_\mu A = \partial_\mu \{L(T)A\}$  for an arbitrary quantity  $A$ .

**Proposition 2** For two arbitrary vector like objects  $A_\mu$ ,  $B_\mu$ ,

$$\frac{A_0}{B_0}, \quad A_i - \frac{A_0}{B_0} B_i \quad (2.24)$$

are scalar like objects.

**Corollary** For a scalar like object  $A$ ,  $D_t A$  where

$$D_t := \frac{1}{\alpha} \frac{\partial}{\partial t} \quad (2.25)$$

is also a scalar like object.

Please notice that  $u_0 = -\alpha + O(\epsilon^2)$ , and that  $\partial_\mu A$  is a vector like object.

**Corollary** For scalar like objects  $A, B$ ,  $D_i(A)B$  where

$$D_i(A) := \partial_i - \frac{\partial_i A}{\dot{A}} \frac{\partial}{\partial t}, \quad (2.26)$$

is also a scalar like object.

Please notice that  $\partial_\mu A, \partial_\mu B$  are vector like objects.

**Corollary** For a scalar like object  $A$ ,

$$\partial_i A + \frac{\dot{A}}{\alpha^2} (v_i + \beta_i) \quad (2.27)$$

is also a scalar like object.

Please notice that the above quantity can be written as  $\partial_i A - (\dot{A}/u_0)u_i$  where  $u_\mu$  is the velocity vector of the total system and that  $\partial_\mu A$  is a vector like object.

**Proposition 3** In the background level, all the evolution equations and all the constraints can be expressed in the form that polynomials of the scalar like objects only are vanishing.

As the proof, we write down the Einstein equations. As for the space-space components of the metric tensor, we use the matrix notation:  $M := (\tilde{\gamma}_{ij})$ ,  $M^{-1} := (\tilde{\gamma}^{ij})$ .  $H$  is the Hubble parameter defined by  $\dot{a}/a$ . The Einstein equations  $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$  give the Hamiltonian constraint

$$\left(\frac{1}{\alpha}H\right)^2 = \frac{\kappa^2}{3}\rho + \frac{1}{24}\text{tr}\left(\frac{1}{\alpha}\dot{M}M^{-1}\frac{1}{\alpha}\dot{M}M^{-1}\right), \quad (2.28)$$

and the evolution equations

$$\frac{1}{\alpha}\frac{\partial}{\partial t}\left(\frac{H}{\alpha}\right) = -\frac{1}{8}\text{tr}\left(\frac{1}{\alpha}\dot{M}M^{-1}\frac{1}{\alpha}\dot{M}M^{-1}\right) - \frac{\kappa^2}{2}(\rho + P), \quad (2.29)$$

$$\frac{1}{\alpha}\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\frac{\partial M}{\partial t}\right) + 3\frac{H}{\alpha}\left(\frac{1}{\alpha}\dot{M}\right) - \frac{1}{\alpha}\dot{M}M^{-1}\frac{1}{\alpha}\dot{M} = 0, \quad (2.30)$$

and the momentum constraint

$$\begin{aligned} 0 &= \frac{1}{2}\frac{\dot{a}}{\alpha}D_i(a)\left(\frac{\alpha}{\dot{a}}\right)\left(M^{-1}\frac{1}{\alpha}\dot{M}\right)_j^i + \frac{1}{2}\left(M^{-1}D_i(a)M \cdot M^{-1}\frac{1}{\alpha}\dot{M}\right)_j^i \\ &\quad - \frac{1}{2}\left[M^{-1}\frac{1}{\alpha}\partial_t\{D_i(a)M\}\right]_j^i + \frac{1}{4}\text{tr}\left(M^{-1}D_j(a)M \cdot M^{-1}\frac{1}{\alpha}\dot{M}\right) \\ &\quad + 2D_j(a)\left(\frac{H}{\alpha}\right) - \kappa^2 h \frac{\alpha}{aH}Z_j, \end{aligned} \quad (2.31)$$

where  $Z_i$  is the scalar like object defined by

$$Z_i := \partial_i a + \frac{\dot{a}}{\alpha^2} (v_i + \beta_i). \quad (2.32)$$

$\square\phi_a - \partial U/\partial\phi_a = S_a$  gives

$$\frac{1}{\alpha} \frac{\partial}{\partial t} \left( \frac{1}{\alpha} \frac{\partial\phi_a}{\partial t} \right) + 3 \frac{H}{\alpha} \frac{\dot{\phi}_a}{\alpha} + \frac{\partial U}{\partial\phi_a} + S_a = 0. \quad (2.33)$$

As for the perfect fluid components,  $\nabla_\mu T_{\alpha 0}^\mu = Q_{\alpha 0}$  and  $\nabla_\mu T_{\alpha i}^\mu = Q_{\alpha i}$  give

$$\frac{1}{\alpha} \dot{\rho}_\alpha = -3 \frac{H}{\alpha} (\rho_\alpha + P_\alpha) + Q_\alpha, \quad (2.34)$$

and

$$\begin{aligned} 0 = & \frac{1}{\alpha a^3} \left[ a^2 h_\alpha \frac{\alpha}{H} Z_{\alpha i} \right] + D_i(a) P_\alpha + h_\alpha a \frac{H}{\alpha} D_i(a) \left( \frac{\alpha}{\dot{a}} \right) \\ & - \frac{1}{a} \frac{\alpha}{H} Q_\alpha Z_i - f_{\alpha i} \end{aligned} \quad (2.35)$$

where  $h_\alpha := \rho_\alpha + P_\alpha$  is the enthalpy of the fluid component  $\alpha$  and  $Z_{\alpha i}$  is the scalar like object defined by

$$Z_{\alpha i} := \partial_i a + \frac{\dot{a}}{\alpha^2} (v_{\alpha i} + \beta_i). \quad (2.36)$$

Then we conclude that all the evolution equations and all the constraints can be written in terms of the scalar like objects only. The evolution equations of  $M$  (2.30) can be solved as

$$M = R_1 \exp \left[ \int_{t_0} dt \frac{\alpha}{a^3} R_2 \right], \quad (2.37)$$

where  $R_1, R_2$  are the  $3 \times 3$  time independent matrices depending on  $\mathbf{x}$ :  $R_1$  is unimodular symmetric,  $R_2$  is traceless and  $R_1 R_2$  is symmetric. [9] By using (2.37), the term in (2.28)(2.29) can be written as

$$\frac{1}{4} \text{tr} \left( \frac{1}{\alpha} \dot{M} M^{-1} \frac{1}{\alpha} \dot{M} M^{-1} \right) = \frac{c_R}{a^6} \quad (2.38)$$

where

$$c_R := \frac{1}{4} \text{tr}(R_2^2). \quad (2.39)$$

We consider the perturbation. We assume that the arbitrary background quantity  $A$  depends not only on  $(t, \mathbf{x})$ , but also on  $\lambda$  which characterizes the perturbation. We can Taylor expand  $A$  around  $\lambda = 0$  as

$$A(\lambda = 1) = \sum_{k=0}^{\infty} \frac{1}{k!} \left. \frac{d^k A(\lambda)}{d\lambda^k} \right|_{\lambda=0}, \quad (2.40)$$

where  $A(\lambda = 1)$  is a full nonlinear quantity. We can identify

$$\left. \frac{d^k A(\lambda)}{d\lambda^k} \right|_{\lambda=0} \leftrightarrow \delta^k A, \quad (2.41)$$



where  $\delta^k A$  is the  $k$ -th order perturbation of  $A$ . The gauge transformation of the background quantity  $A$  is defined by

$$A(\lambda, \mu) = \exp [\mu L(T)] A(\lambda, \mu = 0), \quad (2.42)$$

where  $L(T)$  is the Lie derivative generated by the infinitesimal displacement  $T := T^\mu \partial_\mu$ ,  $A(\lambda, \mu = 0)$  is the quantity before the gauge transformation and  $A(\lambda, \mu = 1)$  is the quantity after the gauge transformation. This expression (2.42) is a solution of the differential equation

$$\frac{d}{d\mu} A = L(T) A, \quad (2.43)$$

which we use instead of the solution (2.42) from now on. By differentiating (2.43) with respect to  $\lambda$ , we get

$$\frac{d}{d\mu} \frac{dA}{d\lambda} = L \left( \frac{dT}{d\lambda} \right) A + L(T) \frac{dA}{d\lambda}, \quad (2.44)$$

since not only the background quantity  $A$  but also the infinitesimal displacement generating the Lie derivative  $T$  depends upon  $\lambda$ . In general, the gauge transformation of the  $\lambda$  derivative of  $A$  contain the Lie derivatives generated by  $d^k T / d\lambda^k$ . But we can make a new quantity  $B$  by combining the  $\lambda$  derivatives of the background quantity appropriately, so that

$$\frac{d}{d\mu} B = L(T) B \quad (2.45)$$

which does not contain the Lie derivatives generated by  $d^k T / d\lambda^k$  ( $k = 1, 2, \dots$ ) can hold. So we can put the definition as follows.

**Definition** We call a quantity  $B$  which has the gauge transformation (2.45) the background like object.

Any background like object is a gauge invariant quantity with respect to all the infinitesimal gauge transformation satisfying  $T(\lambda = 0) = 0$ . We can prove the following proposition.

**Proposition 4** Let  $A, B$  be the scalar like objects and the background like objects. Then  $D(A)B$  where

$$D(A) := \frac{d}{d\lambda} - \frac{dA}{d\lambda} \frac{1}{A} \frac{d}{dt}, \quad (2.46)$$

is also a scalar like object and a background like object.

**Corollary** Under the assumptions in the previous proposition,  $D(A)^n B$  ( $n = 1, 2, \dots$ ) are also scalar like objects and background like objects.

For the proofs, please see our previous paper. [9]

**Proposition 5** All the perturbation equations of the evolution equations and the constraints can be expressed in the form that the polynomials of the quantities which are the scalar like objects and the background like objects are vanishing. That is, all the perturbation equations of the evolution equations and the constraints can be expressed in a manifestly gauge invariant manner.

The perturbation equations can be obtained by operating  $D(S)$  where  $S$  is an arbitrary scalar like object on (2.28) (2.29) (2.30) (2.31) (2.33) (2.34) (2.35) which are written in terms of the scalar like objects. Since a scalar like object operated  $D(S)$  on is a scalar like object and a background like object, the assertion of the proposition 5 follows. If we want to move the time derivative to the outermost position, you can use

$$\left[ D(S), \frac{1}{\alpha} \frac{\partial}{\partial t} \right] = -\frac{\dot{S}}{\alpha} D(S) \left( \frac{\alpha}{\dot{S}} \right) \frac{1}{\alpha} \frac{\partial}{\partial t}, \quad (2.47)$$

where  $[A, B] := AB - BA$ .

## 2.2 the junction conditions

In the early universe, there exist periods when the equation of state changes quite rapidly, such as the slow rolling-oscillatory transition and the reheating transition [4]. As the zeroth order approximation, it is appropriate to treat these transitions by connecting two spacetimes which have different equations of state by the metric junction formalism [23]. These transition hypersurfaces are defined by the particular equations; for the slow rolling-oscillatory transition,  $H/\alpha = m$ , and for the reheating transition  $H/\alpha = \Gamma$  where  $m, \Gamma$  are the mass, the decay constant of the scalar field, respectively. Motivated by the above point, we extend the metric junction theory across the spacelike hypersurface defined by  $C = 0$  where  $C$  is the scalar like object, within the framework of the full nonlinear perturbation theory in the leading order of the gradient expansion. In the appendix B, we formulate the metric junction in the linear perturbation theory in the long wavelength limit, and mention its consistency with the full nonlinear theory.

We consider a 4 dimensional spacetime  $\mathcal{M}$  and a 3 dimensional hypersurface  $\Sigma$ .  $\Sigma$  separates  $\mathcal{M}$  into two region:  $\mathcal{M}_+$  which is the future of  $\Sigma$  and  $\mathcal{M}_-$  which is the past of  $\Sigma$ . The hypersurface  $\Sigma$  is parametrized by the intrinsic coordinate  $y^i$  ( $i = 1, 2, 3$ ) as

$$x_{\pm}^0 = t_{\times} + \delta Z_{\pm}(y), \quad (2.48)$$

$$x_{\pm}^i = y^i + \delta Z_{\pm}^i(y), \quad (2.49)$$

where  $x_{\pm}^{\mu}$  are spacetime coordinates in the region  $\mathcal{M}_{\pm}$ , respectively and  $t_{\times}$  is a constant common to  $\mathcal{M}_{\pm}$ . From now on, we omit index  $\pm$ . The gauge transformation of  $\delta Z$ ,  $\delta Z^i$  are given by

$$L(T)\delta Z = -T^0, \quad (2.50)$$

$$L(T)\delta Z^i = -T^i. \quad (2.51)$$

**Proposition 6** Let  $A_{\mu}, B_{\mu}$  be vector like objects. Then  $A_0 \partial_i \delta Z + (A_0/B_0) B_i$  is the scalar like object.

**Corollary**  $\phi_i := \alpha \partial_i \delta Z + (\alpha/\dot{a}) \partial_i a$  is a scalar like object.

In the previous proposition, as  $A_\mu$ ,  $B_\mu$ , please adopt  $u_\mu$ ,  $\partial_\mu a$ , respectively.

As the junction hypersurface, we adopt the hypersurface characterized by  $C = 0$  where  $C$  is a scalar like object. Then we get

$$\partial_i \delta Z = -\frac{\partial_i C}{\dot{C}}, \quad (2.52)$$

which yields

$$\phi_i = -\frac{\alpha}{\dot{C}} D_i(a) C. \quad (2.53)$$

The normal vector  $n_\mu$  of the hypersurface  $\Sigma$  pointing from  $\mathcal{M}_-$  to  $\mathcal{M}_+$  is given by

$$n_\mu = \frac{-\text{sgn}(\dot{C})}{\sqrt{-g^{\rho\sigma} \partial_\rho C \partial_\sigma C}} \partial_\mu C, \quad (2.54)$$

and the tangential vector  $e_i^\mu$  on  $\Sigma$  are given by

$$e_i^\mu = \frac{\partial x^\mu}{\partial y^i} \quad (i = 1, 2, 3). \quad (2.55)$$

Then we get

$$n_\mu n^\mu = -1, \quad n_\mu e_i^\mu = 0. \quad (2.56)$$

We define the intrinsic metric  $q_{ij}$  and the extrinsic curvature  $K_{ij}$  of  $\Sigma$  by

$$q_{ij} := e_i^\mu e_j^\nu (g_{\mu\nu} + n_\mu n_\nu), \quad (2.57)$$

$$K_{ij} := e_i^\mu e_j^\nu \nabla_\mu n_\nu. \quad (2.58)$$

In case of  $\Sigma$  defined by  $C = 0$ , we obtain

$$q_{ij} = \gamma_{ij} + O(\epsilon^2), \quad (2.59)$$

$$K_{ij} = \frac{1}{2\alpha} \dot{\gamma}_{ij} + O(\epsilon^2). \quad (2.60)$$

As for the energy momentum tensor  $T_{\mu\nu}$ , we obtain

$$T_{nn} := n^\mu n^\nu T_{\mu\nu} = \rho + O(\epsilon^2), \quad (2.61)$$

$$T_{ni} := n^\mu e_i^\nu T_{\mu\nu} = -(\rho + P) \frac{\alpha}{\dot{a}} \{Z_i - D_i(C) a\} + O(\epsilon^3), \quad (2.62)$$

$$T_{ij} := e_i^\mu e_j^\nu T_{\mu\nu} = P \gamma_{ij} + O(\epsilon^2). \quad (2.63)$$

We notice that  $q_{ij}$ ,  $K_{ij}$ ,  $T_{nn}$ ,  $T_{ni}$  and  $T_{ij}$  can be written by the scalar like objects only. The junction condition formulated by Israel [23] is given by

$$[q_{ij}]_-^+ = [K_{ij}]_-^+ = [T_{nn}]_-^+ = [T_{ni}]_-^+ = 0, \quad (2.64)$$

where  $[Q]_-^+ := Q_+ - Q_-$ . In our notation, the above junction condition is written by

$$[a]_-^+ = [\tilde{\gamma}_{ij}]_-^+ = \left[ \frac{\dot{a}}{\alpha} \right]_-^+ = \left[ \frac{\dot{\gamma}_{ij}}{\alpha} \right]_-^+ = [\rho]_-^+ = [(\rho + P) \{Z_i - D_i(C) a\}]_-^+ = 0. \quad (2.65)$$

**Proposition 7** In the background level, the metric junction condition can be expressed in terms of the scalar like objects only.

We consider the perturbation of the junction condition. As for the perturbation, the next proposition is essential.

**Proposition 8** Let the matching hypersurface be defined by  $C = 0$  where  $C$  is a scalar like object. For an arbitrary scalar like object  $S$  satisfying  $[S]_{-}^{+} = 0$ ,  $D(C)S$ ,  $D_i(C)S$  are continuous across the matching hypersurface:  $[D(C)S]_{-}^{+} = 0$ ,  $[D_i(C)S]_{-}^{+} = 0$ .

For the proof, please see the appendix A. By applying the above proposition finite times, we obtain the following corollary.

**Corollary** Under the assumption presented by the previous proposition,  $[D(C)^n S]_{-}^{+} = 0$  for an arbitrary natural number  $n$ .

As for  $M := (\tilde{\gamma}_{ij})$ ,  $M$  is solved as (2.37). From (2.65),  $M$  in  $\mathcal{M}_+$  is given by

$$M_+ = R_{1-} \exp \left[ \int_{t_0}^{t_x + \delta Z_-} dt \frac{\alpha}{a^3} R_2 + \int_{t_x + \delta Z_+}^t dt \frac{\alpha}{a^3} R_2 \right], \quad (2.66)$$

where  $R_{1-}$  is  $R_1$  in  $\mathcal{M}_-$ , and  $R_2$  in  $\mathcal{M}_+$  and  $R_2$  in  $\mathcal{M}_-$  is the same  $R_{2+} = R_{2-} =: R_2$ .

As the junction, we consider the transition where the energy  $\rho_{A-}$  transfers into the energy  $\rho_{A+}$  which has the different equation of state from  $\rho_{A-}$ . From (2.65), the energy momentum conservation

$$[\rho_A]_{-}^{+} = 0, \quad (2.67)$$

$$[(\rho_A + P_A)\{Z_{Ai} - D_i(C)a\}]_{-}^{+} = 0, \quad (2.68)$$

must hold. In the above discussion, all the perturbation equations of the metric junction conditions are written in the form that the polynomials of the quantities which are the scalar like objects and the background like objects are vanishing, therefore all the perturbation equations of the metric junction conditions are gauge invariant.

### §3 choice of the independent gauge invariant variables based on the classification of the perturbation solutions into the adiabatic mode and the entropic modes

#### 3.1 the universal adiabatic growing mode

All the evolution equations of the locally homogeneous universe are invariant under the transformation defined by

$$a \rightarrow a\Lambda, \quad (3.1)$$

$$R_2 \rightarrow R_2\Lambda^3, \quad (3.2)$$

$$\alpha \rightarrow \alpha, \quad (3.3)$$

$$W \rightarrow W, \quad (3.4)$$

where  $\alpha$  is the lapse function and  $W$  is an arbitrary scalar quantity.  $\Lambda$  is a time independent function  $\Lambda = \Lambda(\mathbf{x})$ . Taking the variation with respect to  $\lambda$  considering that only  $\Lambda$  is dependent upon  $\lambda$  gives the obvious perturbation solution. We call this solution the universal adiabatic growing mode. [5] For an arbitrary scalar quantity  $S$ , the first order and the second order perturbation solutions of the universal adiabatic growing mode is written as

$$D(a)S = -\frac{\dot{S}}{H} \frac{d}{d\lambda} \ln \Lambda, \quad (3.5)$$

$$D(a)^2 S = -\frac{\dot{S}}{H} \frac{d^2}{d\lambda^2} \ln \Lambda + \frac{1}{H} \frac{d}{dt} \left( \frac{\dot{S}}{H} \right) \left( \frac{d}{d\lambda} \ln \Lambda \right)^2, \quad (3.6)$$

where the first order expression is very familiar in large literature. We call the gauge invariant perturbation variable defined by

$$\zeta_n(S) := D(S)^n \ln a \quad (3.7)$$

the generalized Bardeen parameter induced by the scalar like object  $S$ . When we adopt an arbitrary scalar quantity  $W$  or  $H/\alpha$  as the scalar like object  $S$ , the perturbation solution of the universal adiabatic growing mode is written as a time independent form:

$$\zeta_n(S) = \frac{d^n}{d\lambda^n} \ln \Lambda. \quad (3.8)$$

### 3.2 the adiabatic perturbation variable and the entropic perturbation variables

In order to interpret the physics of the linear cosmological perturbations, the classification into the adiabatic perturbation and the entropic perturbations was often convenient. [17] [18] [19] [2] Therefore the generalization of this classification into higher order perturbations are thought to be useful. So we define the adiabatic perturbation variable and the entropic perturbation variables in the higher order perturbation theory.

**Definition** We call the perturbation variable which does not vanish for the universal adiabatic growing mode the adiabatic perturbation variable. We call the perturbation variable which vanishes for the universal adiabatic growing mode the entropic perturbation variable.

We will present the examples of the adiabatic and the entropic perturbation variables. We assume that  $S$ ,  $S_i$  ( $i = 1, 2$ ) are the scalar like objects such as  $W$ ,  $\dot{W}/\alpha$ ,  $H/\alpha$  where  $W$  is an arbitrary scalar variable. The generalized Bardeen parameter  $\zeta_n(S)$  and  $D(a)^n S$  are adiabatic perturbation variables and  $\zeta_n(S_1) - \zeta_n(S_2)$  and  $D(S_1)^n S_2$  are entropic perturbation variables.

### 3.3 the $N$ philosophy and the $\rho$ philosophy

We call the expressions representing the physical quantities at the final time in terms of those at the initial time the  $S$  formulas. [8] Our final purpose is to construct the  $S$  formulas

of the adiabatic perturbation variable such as the Bardeen parameter  $\zeta_n(\rho) := D(\rho)^n \ln a$ . In the previous subsection, it was shown that the Bardeen parameter  $\zeta_n(\rho)$  is time independent for the universal adiabatic growing mode. Therefore we expect that the formulation in which the difference between the Bardeen parameter at the final time and that at the initial time can be expressed in terms of the entropic perturbation variables may exist. In this subsection, we choose the appropriate set of the entropic perturbation variables and we construct the formulation in which the time change of the Bardeen parameter is brought about by the evolutions of the set of these entropic perturbation variables.

Until now in order to understand the evolutions of linear cosmological perturbations in the universe governed by the multiple component energy densities, the decomposition of the perturbations into the adiabatic component and the entropic components has already been performed.[17] [18] [19] [2] In the nonlinear perturbation theory, the following set of perturbation variables was adopted: as the adiabatic perturbation variable, the Bardeen parameter  $\zeta_n(\rho)$ , and as the entropic perturbation variables, the difference between the generalized Bardeen parameters induced by the energy densities of the different components  $S_n(\rho_A, \rho_B) := \zeta_n(\rho_A) - \zeta_n(\rho_B)$  where the subscripts  $A, B$  represent the different components.[20] Since all the perturbation variables in this formulation [20] are based on perturbations of the logarithm of the scale factor  $N := \ln a$ , we call this formulation the  $N$  philosophy. But in the  $N$  philosophy, it is difficult to write down the evolution equations in terms of the set of variables  $\zeta_n(\rho)$ ,  $S_n(\rho_A, \rho_B)$  in the closed form. From now on, we often consider the matching of the metric across the matching hypersurface defined by  $\rho = \text{const}$ . Across such matching hypersurface, the perturbation variables in the  $N$  philosophy,  $S_n(\rho_A, \rho_B)$  jump by finite values.

In order to solve the defects in the  $N$  philosophy, we propose the new set of the perturbation variables. We choose

$$D\left(\frac{H}{\alpha}\right)^n \ln a, \quad D\left(\frac{H}{\alpha}\right)^n s_A, \quad (3.9)$$

where  $s_A := \rho_A/\rho$ , as the adiabatic perturbation variable and the entropic perturbation variables, respectively.  $s_A$  satisfies  $\sum_A s_A = 1$ . As for the new set of the perturbation variables, no finite jumps do not exist across the slow rolling-oscillatory transition  $H/\alpha = m$  where  $m$  is the mass of the scalar field and across the reheating transition  $H/\alpha = \Gamma$  where  $\Gamma$  is the decay constant of the scalar field. In order to avoid the calculational complexity, we assume that  $c_R = 0$ , since we can neglect the second term of the right hand side of the Hamiltonian constraint (2.28) with (2.38) because of the rapid growth of the scale factor  $a$  during the inflationary expansion of the universe. In the condition  $c_R = 0$ , the matching conditions of the slow rolling-oscillatory transition, of the reheating transition are reduced into  $\rho - 3m^2/\kappa^2 = 0$ ,  $\rho - 3\Gamma^2/\kappa^2 = 0$ , respectively. Under the simplification of  $c_R = 0$ , the set of the perturbation variables which we adopted in (3.9) is reduced into

$$D(\rho)^n \ln a, \quad D(\rho)^n s_A. \quad (3.10)$$

Since these perturbation variables are continuous across the matching hypersurface defined by  $\rho = \text{const}$ , we only have to concentrate on solving the evolution equations of these perturbation variables. Since all these variables are defined by  $D(\rho)$ , we call the use of these variables presented in (3.10) the  $\rho$  philosophy.

We will give the evolution equations of the perturbation variables (3.10). For simplicity, we assume that the multiple components do not interact and that  $\rho_A$  obeys  $d\rho_A/dN = -g_A\rho_A$  where  $g_A$  will be called the  $g$  factor from now on. When the  $\alpha$  component is the perfect fluid with  $w_\alpha := P_\alpha/\rho_\alpha$ , its  $g$  factor is given by  $g_\alpha = 3(1 + w_\alpha)$ . When the  $a$  component is the slow rolling massive scalar field with mass  $m_a$ , its  $g$  factor is given by  $g_a = 2m_a^2/\kappa^2\rho$ . Since it was shown that the oscillatory massive scalar field can be approximated by the perfect fluid with  $w_\alpha = 0$  [3] [6] [7] [8], we can use the perfect fluid with  $g_\alpha = 3$  instead of the oscillatory massive scalar field. The evolution equations of  $N := \ln a$ ,  $s_A$  are given by

$$\frac{d}{d\rho}N = -\frac{1}{\rho s}, \quad (3.11)$$

$$\frac{d}{d\rho}s_A = \frac{1}{\rho} \left( -s_A + \frac{g_A s_A}{s} \right), \quad (3.12)$$

where  $s := \sum_B g_B s_B$ . We choose the total energy density  $\rho$  as the evolution parameter instead of the cosmic time  $t$ , and the right hand sides of (3.11), (3.12) are written by  $\rho$ ,  $s_A$  only. The evolution equations of the perturbation variables in the  $\rho$  philosophy are given by operating  $D(\rho)$  finite times on (3.11), (3.12). In this case, it is important to notice that  $D(\rho)$  and  $d/d\rho$  are commutative since  $d/dt$  and  $d/d\lambda$  are commutative.

In the  $\rho$  philosophy, all the perturbation variables are continuous across the matching hypersurface defined by  $\rho = \text{const}$  because of the proposition 8, and we can easily derive the evolution equations of the perturbation variables. Since the  $\rho$  philosophy is superior to the  $N$  philosophy because of the above two reasons, we will adopt the  $\rho$  philosophy from now on.

## §4 the non-Gaussianities of the nonlinear cosmological perturbations

In this section, we discuss the non-Gaussianities generated in several cosmological models. The non-Gaussianities are measured by the  $f_{NL}$  parameter.[35] It is assumed that the logarithm of the scale factor  $N := \ln a$  is given by the function of the energy density  $\rho$  as the evolution parameter and of the solution constants. We only consider the models where the origins of the cosmological perturbations are in the quantum fluctuations of the scalar fields  $\phi_a$  in the inflationary universe. The statistical mean values of the perturbation amplitudes are given by

$$\left\langle \left\langle \frac{d\phi_a(0)}{d\lambda} \frac{d\phi_b(0)}{d\lambda} \right\rangle \right\rangle \sim H^2 \delta_{ab}, \quad (4.1)$$

where  $\phi_a(0)$  is the expectation value of the scalar field  $\phi_a$  at the first horizon crossing and  $H$  is the Hubble parameter at the first horizon crossing. In this case, the solution constants are given by the set of the expectation values of the scalar fields at the first horizon crossing  $\{\phi_a(0)\}$ . In this case, using the logarithm of the scale factor  $N = N(\rho, \phi_1(0), \phi_2(0), \dots)$  the non-Gaussianity parameter  $f_{NL}$  is defined by

$$f_{NL} := \frac{N_{ab} N^a N^b}{(N_c N^c)^2}, \quad (4.2)$$

$$N_a := \frac{\partial}{\partial \phi_a(0)} N, \quad N_{ab} := \frac{\partial^2}{\partial \phi_a(0) \partial \phi_b(0)} N. \quad (4.3)$$

[35]  $N_a$ ,  $N_{ab}$  are defined as the coefficients given when we expand the gauge invariant adiabatic perturbation variables  $D(\rho) \ln a$ ,  $D(\rho)^2 \ln a$  with respect to  $d\phi_a(0)/d\lambda$ , respectively;

$$D(\rho) \ln a = \sum_a N_a \frac{d\phi_a(0)}{d\lambda}, \quad (4.4)$$

$$D(\rho)^2 \ln a = \sum_{ab} N_{ab} \frac{d\phi_a(0)}{d\lambda} \frac{d\phi_b(0)}{d\lambda}. \quad (4.5)$$

We assumed that the more than second order perturbations of  $\phi_a$  at the initial time are all vanishing;  $d^n \phi_a(0)/d\lambda^n = 0$  ( $n \geq 2$ ).

Since the cosmological perturbations have the origin in the scalar field fluctuations in the de Sitter stage, the deviations of the spectral indices of the Bardeen parameter  $\zeta_1(\rho)$  from the scale invariance  $d \ln \langle \zeta_1(\rho) \rangle / d \ln k$  are suppressed by the slow rolling parameter. Since the first horizon crossing is defined by the relation

$$k = aH = e^{N_*} \frac{\kappa}{\sqrt{3}} \rho(0)^{1/2}, \quad (4.6)$$

where  $N_*$ ,  $\rho(0)$  is the logarithm of the scale factor, energy density at the first horizon crossing time, respectively, we obtain

$$d \ln k = \left\{ 1 + \frac{1}{2} \frac{1}{\rho(0)} \frac{d\rho(0)}{dN_*} \right\} dN_*. \quad (4.7)$$

The slow rolling phase is characterized by the smallness of the  $g$  factors of the scalar fields  $\phi_a(0)$  whose size is bounded by  $\delta_S$  a small constant characterizing the slow rolling of the scalar fields;  $|g_a| \leq \delta_S$  where  $g_a$  is defined by the evolution equations of the energy densities of the scalar field  $\phi_a$ ;  $\rho_a$  at the first horizon crossing:  $d\rho_a(0)/dN_* = -g_a \rho_a(0)$ . In all the cases which we consider, the following evaluations hold:  $\partial \ln \langle \zeta_1^2(\rho) \rangle / \partial \rho_a(0) \sim 1/\rho_a(0)$ . By using the above properties, we can conclude that the Bardeen parameter in the first order perturbation theory  $\zeta_1(\rho)$  has almost complete scale invariance:

$$\frac{d}{d \ln k} \ln \langle \zeta_1^2(\rho) \rangle \sim \delta_S. \quad (4.8)$$

Then we concentrate on the non-Gaussianity parameter  $f_{NL}$  from now on.

Except for the cases where the large  $f_{NL}$  is generated discontinuously on the transition hypersurface such as the inhomogeneous end of the inflation [28] [29] and the modulated reheating [30] [31], different two cases have been discussed. One is the curvaton scenario [32] and the other is the vacuum dominated two scalar fields [33] [34]. We discuss that the mechanism which generates the large  $f_{NL}$  continuously in the above two different cases can be explained from the three common viewpoints which will be presented in the subsection 2 of this section. In this section, we use two strong methods; the exponent evaluation method and the  $\rho$  philosophy. By the exponent evaluation method, it becomes possible to evaluate the order of  $f_{NL}$  analytically in the wide ranges of parameters. In the  $\rho$



philosophy, the time evolutions of the above two systems are traced using the logarithm of the scale factor  $N$  as the adiabatic independent variable,  $s_2$  which implies the ratio of the energy density of the component which does not govern the energy density of the universe as the entropic independent variable, and the energy density  $\rho$  as the evolution parameter. The  $\rho$  philosophy makes the instant when  $f_{NL}$  grows large and the length of the period when the large  $f_{NL}$  continues clear.

## 4.1 the exponent evaluation method

In many papers, the calculations of the non-Gaussianity parameter  $f_{NL}$  were performed: for the inhomogeneous end of the inflation [28] [29], for the modulated reheating [31], for the curvaton [32] and for the vacuum dominated inflation represented by the hybrid inflation [33] [34]. In many papers so far, the calculations of  $f_{NL}$  were performed numerically and in the extreme situations where only one factor is concerned the analytic formulas of  $f_{NL}$  were given. In this subsection, we present the exponent evaluation method by which it becomes possible to give the analytic order estimates of  $f_{NL}$  in the wide range including cases where more than two factors are concerned.

We explain the exponent evaluation method by adopting the inhomogeneous end of the inflation [28] [29] as an example. We consider the two scalar fields  $\phi_1, \phi_2$  governed by the vacuum dominated potential given by

$$U = U_0 + \sum_{a=1}^2 \frac{1}{2} \eta_a \phi_a^2, \quad (4.9)$$

where  $\eta_a$  ( $a = 1, 2$ ) are negative and  $U_0$  is the large constant compared with the terms quadratic with the scalar fields. Under the approximation where the vacuum energy  $U_0$  is the dominant contribution of the energy density  $\rho$  and where the scalar fields  $\phi_a$  ( $a = 1, 2$ ) are slow rolling on the potential, the evolutions of  $\phi_a$  ( $a = 1, 2$ ) are given by

$$\phi_a = \phi_a(0) \exp \left[ -\frac{\eta_a}{\kappa^2 U_0} N \right]. \quad (4.10)$$

For simplicity,  $\eta_a$  ( $a = 1, 2$ ) are assumed to be  $a$  independent:  $\eta_a = \eta$ . We assume that the inflation ends in the bifurcation set defined by

$$\sum_a \gamma_a \phi_a^2 = \sigma^2. \quad (4.11)$$

On the bifurcation set, the waterfall field which interacts with the inflatons  $\phi_a$  ( $a = 1, 2$ ) gets the negative mass and the large vacuum energy  $\sim U_0$  is transferred into the oscillation energy of the waterfall field and into the radiation energy which interacts with the waterfall field. Also in the situation where along the curve defined by (4.11) the deep ditch of the potential exists, the inflation ends on the curve (4.11). The logarithm of the scale factor  $N$  at which the scalar fields reach the bifurcation set (4.11) is given by

$$N = \frac{\kappa^2 U_0}{2\eta} \ln \left[ \frac{1}{\sigma^2} \sum_a \gamma_a \phi_a^2(0) \right]. \quad (4.12)$$

In order that the inflation can solve the horizon problem, we assume that  $\alpha := \kappa^2 U_0 / 2\eta = 10^2$ . The non-Gaussianity parameter  $f_{NL}$  is calculated as

$$f_{NL} = \frac{1}{\alpha} \left( \frac{A_1 A_3}{2A_2^2} - 1 \right) =: \frac{1}{\alpha}(p - 1), \quad (4.13)$$

where

$$A_n := \sum_{a=1}^2 \gamma_a^n \phi_a^2(0). \quad (4.14)$$

From now on, we neglect numerical factors of order unity without mentioning. By setting

$$\gamma_r := \frac{\gamma_2}{\gamma_1} = 10^k, \quad \phi_r := \frac{\phi_2(0)}{\phi_1(0)} = 10^{-l}, \quad (4.15)$$

we obtain

$$p = \frac{(1 + 10^{k-2l})(1 + 10^{3k-2l})}{(1 + 10^{2k-2l})^2}. \quad (4.16)$$

The exponent evaluation method gives

$$k < \frac{2}{3}l \quad p = 1, \quad (4.17)$$

$$\frac{2}{3}l < k < l \quad p = 10^{3k-2l}, \quad (4.18)$$

$$l < k < 2l \quad p = 10^{-k+2l}, \quad (4.19)$$

$$2l < k \quad p = 1. \quad (4.20)$$

As an example of application of the exponent evaluation method, we adopt  $l < k < 2l$ . Since  $l < k < 2l$ , we obtain

$$1 + 10^{k-2l} \sim 1, \quad (4.21)$$

$$1 + 10^{3k-2l} \sim 10^{3k-2l}, \quad (4.22)$$

$$1 + 10^{2k-2l} \sim 10^{2k-2l}, \quad (4.23)$$

and get

$$p = \frac{1 \cdot 10^{3k-2l}}{(10^{2k-2l})^2} = 10^{-k+2l}. \quad (4.24)$$

$p$  takes the maximum at  $k = l$ :  $p = 10^l$ , then  $f_{NL} = 10^{l-2}$ . In the exponent evaluation method, we take only the term which has the largest exponent in the polynomial constructed by several  $10^M$  type terms.

As the second example of application of the exponent evaluation method, we consider the modulated reheating. [30] [31] The scalar field  $\phi_1$  on the potential  $U = m^2 \phi_1^2 / 2$  causes the chaotic inflation:

$$N = \frac{\kappa^2}{4} \{ \phi_1^2(0) - \phi_1^2 \}. \quad (4.25)$$

When  $H^2 = m^2$  where  $H$  is the Hubble parameter, the scalar field  $\phi_1$  begins to oscillate and behaves like a dust fluid [3] [6] [7] [8]:

$$N = \frac{\kappa^2}{4} \phi_1^2(0) - \frac{3}{2} - \frac{1}{3} \ln \left( \frac{\kappa^2 \rho}{3m^2} \right). \quad (4.26)$$

When  $H^2 = \Gamma^2$  where  $\Gamma$  is the decay constant of the scalar field  $\phi_1$ , the scalar field oscillation is transformed into the radiation fluid:

$$N = \frac{\kappa^2}{4}\phi_1^2(0) - \frac{3}{2} - \frac{1}{3}\ln\frac{\Gamma^2}{m^2} - \frac{1}{4}\ln\left(\frac{\kappa^2\rho}{3\Gamma^2}\right). \quad (4.27)$$

In the modulated reheating, we consider that the decay constant of the first scalar field  $\phi_1$ ;  $\Gamma$  is the function of the second scalar field  $\phi_2$  as  $\Gamma = \alpha_d\phi_2^n(0)$  where  $\alpha_d$  is a constant and  $n$  is an integer. In this case,  $N$  takes the form as

$$N = \kappa^2\phi_1^2(0) + \ln\phi_2(0), \quad (4.28)$$

up to the  $\rho$  dependent part which does not contribute the Bardeen parameters  $\zeta_n(\rho)$ . In order that the  $\phi_1$  inflation can solve the horizon problem, we assume that  $\phi_1(0) = 10/\kappa$ . We assume that the second scalar field  $\phi_2(0)$  takes a small value as  $\phi_2(0) = 10^{-l}/\kappa$ . The non-Gaussianity parameter  $f_{NL}$  is calculated as

$$f_{NL} = \frac{10^2 - 10^{4l}}{(10^2 + 10^{2l})^2}. \quad (4.29)$$

The exponent evaluation method gives

$$l < \frac{1}{2} \quad f_{NL} = 10^{-2}, \quad (4.30)$$

$$\frac{1}{2} < l < 1 \quad f_{NL} = -10^{-4(l-1)}, \quad (4.31)$$

$$1 < l \quad f_{NL} = -1, \quad (4.32)$$

When the second scalar field  $\phi_2$  takes a very small value, a significant non-Gaussianity  $f_{NL}$  is generated.

## 4.2 the non-Gaussianities induced by the entropic perturbation of the component which does not govern the cosmic energy density

In this subsection, we investigate the mechanism which triggers the large  $f_{NL}$  in the different two models 1, 2 where the large  $f_{NL}$  can be generated continuously. The model 1 is the radiation-dust system and the non-Gaussianity  $f_{NL}$  in the model 1 was investigated in the paper [32] in the context of the curvaton scenario. The model 2 is the vacuum dominated two scalar fields and the non-Gaussianity  $f_{NL}$  in the model 2 was investigated in the papers [33] [34] in the context of the hybrid inflation. Although the model 1 and the model 2 are quite different apparently, we consider the mechanisms which generate the large  $f_{NL}$ 's in the model 1 and in the model 2 are completely the same. It is assumed that the inflation sufficient to solve the horizon problem  $N \sim 10^2$  is brought about by the first scalar field  $\phi_1$ . Under this assumption, the common points are summarized in the following three points:

- A. The expectation value of the second scalar field  $\phi_2$ ;  $\phi_2(0)$  is very small compared with that of the first scalar field  $\phi_1$ ;  $\phi_1(0)$  at the first horizon crossing.

- B. The component which originates from the second scalar field  $\phi_2(0)$  does not govern the cosmic energy density  $\rho$ .
- C. The  $g$  factor of the component originating from the second scalar field  $\phi_2(0)$ ;  $g_2$  is smaller than the  $g$  factor of the component governing the cosmic energy density  $\rho$ ;  $g_1$ .

The condition  $A$  guarantees that the contribution from the second scalar field  $\phi_2(0)$  to the Bardeen parameters  $\zeta_n(\rho)$  ( $n = 1, 2, \dots$ ) is large. For example, we assume that  $N$  can be written as the sum of the  $\phi_i(0)$  dependent parts;  $N = \sum_i f_i(\phi_i(0))$  and that each  $f_i$  is the power or the logarithm of  $\phi_i(0)$ . Then the contribution from  $\phi_2(0)$  to  $\zeta_1(\rho)$  is proportional to  $\partial N / \partial \phi_2(0) \sim f_2(\phi_2(0)) / \phi_2(0)$  which is quite large when  $\phi_2(0)$  is very small even if the contribution from  $\phi_2(0)$  to  $N$ ;  $f_2(\phi_2(0))$  is quite small. Owing to the condition  $A$ , it becomes possible that the entropic perturbation of the component which does not govern the cosmic energy density  $\rho$  brings about significant contributions to the Bardeen parameters  $\zeta_n(\rho)$  ( $n = 1, 2, \dots$ ) without contributing  $N$ . From the structure of the evolution equation of  $N := \ln a$  (3.11) and that of  $s_A$  (3.12), the growth of the Bardeen parameters  $\zeta_n(\rho) := D(\rho)^n N$  are governed by the entropic perturbation  $D(\rho)^k s_A$  of the component  $A$  whose ratio of the energy density  $s_A$  is very small  $|s_A| \ll 1$ . In addition, when  $|s_A| \ll 1$ , the entropy perturbation  $D(\rho)^k s_A$  can grow or decrease rather rapidly. Therefore the condition  $B$  requires the existence of such component. Owing to the condition  $C$ ;  $g_2 < g_1$ , the contribution of  $s_2$  to  $N$  is monotonically increasing for the time evolution when the evolution parameter  $\rho$  becomes decreasing, therefore the contribution from  $D(\rho)^k s_2$  to the Bardeen parameter  $\zeta_n(\rho) := D(\rho)^n N$  is also increasing.

Since  $g_2 < g_1$  from the condition  $C$ , the ratio of the energy density of the second component  $s_2$  grows compared with that of the first component  $s_1$ . Since the second component  $s_2$  begins to dominate the cosmic energy density  $\rho$  soon, the period when the condition  $B$  is satisfied is only the early period of time from the beginning. So the large  $f_{NL}$  is realized only transiently in this period.

Both the model 1 and the model 2 share the above three properties. The model 1 is treated as the case without the cosmological term in section 4.2.1 and the model 2 is treated as the case with the cosmological term in section 4.2.2.

In the rest of this subsection 4.2, we analyse the concrete physical systems using  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$  as the independent perturbation variables and  $\rho$  as the evolution parameter. This scheme which we proposed as the  $\rho$  philosophy in subsection 3.3 is supported by the results of sections 2, 3. Subsection 2.1 guarantees that these independent perturbation variables  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$  are gauge invariant perturbation variables. Subsection 3.2 states that  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$  can be regarded as the adiabatic perturbation, the entropic perturbations in the higher order perturbation, respectively. This A/E interpretation is useful when we interpret the time evolutions of the concrete physical systems. Since  $D(\rho)\rho = 0$ , that is  $D(\rho)$  can be interpreted as the partial derivative with respect to  $\lambda$  with  $\rho$  fixed, the  $\rho$  dependences of  $\ln a$ ,  $s_A$  are directly reflected to the  $\rho$  dependences of  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$ . This fact makes the calculations and the interpretations of the time evolutions of the perturbation variables transparent. By the  $\rho$  philosophy which we explained in the above, in the rest of this subsection 4.2 we clarify that the entropy perturbation  $D(\rho)^n s_2$  supported by the energetically subdominant component  $\rho_2$  makes the

adiabatic perturbation  $D(\rho)^n \ln a$  and the non-Gaussianity  $f_{NL}$  grow considerably under the conditions that the  $g$  factor of this energetically subdominant component  $\rho_2$  is smaller than the  $g$  factor of the energetically dominant component  $\rho_1$  and that the subdominant component  $\rho_2$  is supported by the extremely small scalar field expectation value.

#### 4.2.1 the case without the cosmological term

We consider the two component system. We assume that the  $g$  factor of the component  $\rho_A$  ( $s_A$ ) is  $g_A$ . Assuming that  $|s_2| \ll 1$  and linearizing (3.11)(3.12) with respect to  $s_2$ , we obtain

$$\frac{d}{d\rho}s_2 = \frac{1}{\rho} \left( -1 + \frac{g_2}{g_1} \right) s_2, \quad (4.33)$$

$$\frac{d}{d\rho}N = -\frac{1}{\rho g_1} + \frac{1}{\rho g_1} \left( -1 + \frac{g_2}{g_1} \right) s_2, \quad (4.34)$$

whose solution is given by

$$s_2 = \frac{\rho_2(1)}{\rho_1(1)} \left( \frac{\rho}{\rho_1(1)} \right)^{-1+g_2/g_1}, \quad (4.35)$$

$$N = N(1) - \frac{1}{g_1} \ln \frac{\rho}{\rho_1(1)} + \frac{1}{g_1} \frac{\rho_2(1)}{\rho_1(1)} \left( \frac{\rho}{\rho_1(1)} \right)^{-1+g_2/g_1}, \quad (4.36)$$

where the  $g$  factors are assumed to be constant and  $X(1)$  implies the physical quantity  $X$  at an initial time. When  $g_2 < g_1$ ,  $s_2$  and the contribution to  $N$  from the  $\rho_2(1)$  dependent term increase and they are not bounded for the time evolution  $\rho \rightarrow 0$ . When  $g_2 > g_1$ ,  $s_2$  and the contribution to  $N$  from the  $\rho_2(1)$  dependent term decrease for the time evolution.

When  $g_2 < g_1$ , the ratio of the energy density  $s_2$  increases and reaches almost unity. In this case, the evolution equations (3.11) (3.12) give

$$\frac{d}{d\rho}s_1 = \frac{1}{\rho} \left( -1 + \frac{g_1}{g_2} \right) s_1, \quad (4.37)$$

$$\frac{d}{d\rho}N = -\frac{1}{\rho g_2} + \frac{1}{\rho g_2} \left( -1 + \frac{g_1}{g_2} \right) s_1, \quad (4.38)$$

by linearizing (3.11) (3.12) with respect to  $s_1$  assuming that  $|s_1| \ll 1$ . The solution is given by

$$s_1 = \frac{\rho_1(1)}{\rho_2(1)} \left( \frac{\rho}{\rho_2(1)} \right)^{-1+g_1/g_2}, \quad (4.39)$$

$$N = N(1) - \frac{1}{g_2} \ln \frac{\rho}{\rho_2(1)} + \frac{1}{g_2} \frac{\rho_1(1)}{\rho_2(1)} \left( \frac{\rho}{\rho_2(1)} \right)^{-1+g_1/g_2}. \quad (4.40)$$

When  $g_2 < g_1$ ,  $s_1$  and the contribution to  $N$  from the  $\rho_1(1)$  dependent term decrease for the time evolution  $\rho \rightarrow 0$ .

In the above, the independent variables  $\ln a$ ,  $s_A$  are written as functions of  $\rho$  as the evolution parameter. By operating  $D(\rho)$  derivatives on expressions of  $\ln a$ ,  $s_A$ , we can obtain

the expressions of the A/E perturbation variables  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$  in the form of the functions of  $\rho$ . Please notice that the  $D(\rho)$  derivative can be regarded as the partial derivative with respect to  $\lambda$  with the evolution parameter  $\rho$  fixed. Therefore the  $\rho$  dependences of  $\ln a$ ,  $s_A$  completely corresponds with the  $\rho$  dependences of  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$ . In the above calculations and discussions, we can see that the entropic perturbation  $D(\rho)^n s_2$  of the energetically subdominant component  $\rho_2$  with  $g$  factor smaller than the  $g$  factor of the energetically dominant component  $\rho_1$  makes the adiabatic perturbation variable  $D(\rho)^n \ln a$  grow transiently while the energy density ratio  $s_2$  is increasing.

When we calculate the non-Gaussianity  $f_{NL}$ , the expressions of  $N_a$ ,  $N_{ab}$  are necessary.  $N_a$ ,  $N_{ab}$  are defined as the coefficients of the gauge invariant adiabatic perturbation variables  $D(\rho) \ln a$ ,  $D(\rho)^2 \ln a$  given when  $D(\rho) \ln a$ ,  $D(\rho)^2 \ln a$  are expanded with respect to  $d\phi_a(0)/d\lambda$ , respectively. The subscript  $a$  implies the  $\partial/\partial\phi_a(0)$  derivative with  $\rho$  fixed. The  $\rho$  philosophy proposed in subsection 3.3 makes the calculations and the interpretations of the time evolutions of the non-Gaussianity  $f_{NL}$  as well as the A/E perturbation variables  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$  more simple and more transparent.

We apply the above results to the following concrete situation. Before  $N = N(1)$ ,  $\rho_1$  causes the chaotic inflation  $N(1) \sim 10^2$ , and at  $N = N(1)$  decays into the radiation. After  $N = N(1)$ ,  $\rho_1$  is radiation ( $g_1 = 4$ ). After  $N = N(1)$ ,  $\rho_2$  is still in the slow rolling phase. Because we assume that  $m_2 \ll m_1$ ,  $\phi_2$  hardly moves from the initial value  $\phi_2(0)$  ( $g_2 = 0$ ). At  $\kappa^2 \rho/3 = m_2^2$ , the slow rolling phase of  $\rho_2$  ends and begins to oscillate. After  $\kappa^2 \rho/3 = m_2^2$ ,  $\rho_2$  behaves like dust fluid ( $g_2 = 3$ ). [3] [6] [7] [8] Then applying (4.36) to the above situation yields

$$N = N(1) + \frac{1}{4} \ln \frac{\rho_1(1)}{\rho} + \frac{1}{8} m_2^2 \phi_2^2(0) \left( \frac{\kappa^2}{3m_2^2} \right)^{3/4} \frac{1}{\rho^{1/4}} \frac{1}{(1 - \kappa^2 \phi_2^2(0)/6)^{3/4}}. \quad (4.41)$$

This solution is simplified into the model given by

$$N = \kappa^2 \phi_1^2(0) + \kappa^2 \alpha(\rho) \phi_2^2(0), \quad (4.42)$$

up to the  $\rho$  dependent part which is not related with the Bardeen parameters  $\zeta_n(\rho)$ , where  $\alpha(\rho)$  is a function of  $\rho$  and increases for the time evolution  $\rho \rightarrow 0$ . All the numerical coefficients of order unity are dropped without mentioning from now on. We assume that  $\phi_1$  causes the inflation enough to solve the horizon problem and that  $\phi_2$  is a small field enough to contribute to the Bardeen parameters  $\zeta_n(\rho)$  sufficiently:

$$\phi_1(0) = \frac{1}{\kappa} 10, \quad \phi_2(0) = \frac{1}{\kappa} 10^{-l}, \quad (4.43)$$

where  $l$  is a positive number.  $\alpha(\rho)$  is written by  $\alpha(\rho) = 10^k$  where  $k$  increases for the time evolution  $\rho \rightarrow 0$ . Since we adopt the approximation that  $\rho_2$  ( $s_2$ ) does not govern the cosmic energy density  $\rho$ , we obtain  $k < 2l$ . The non-Gaussianity parameter  $f_{NL}$  of the model (4.42) is calculated as

$$f_{NL} = \frac{1}{10^2} \frac{1 + 10^{3k-2l-2}}{(1 + 10^{2k-2l-2})^2}. \quad (4.44)$$

The exponent evaluation method yields

$$k < \frac{2(l+1)}{3} \quad f_{NL} = 10^{-2}, \quad (4.45)$$

$$\frac{2(l+1)}{3} < k < l+1 \quad f_{NL} = 10^{3k-2l-4}, \quad (4.46)$$

$$l+1 < k < 2l \quad f_{NL} = 10^{-k+2l}. \quad (4.47)$$

For  $2(l+1)/3 < k < l+1$ ,  $f_{NL}$  increases and reaches the maximum  $f_{NL} = 10^{l-1}$  at  $k = l+1$ . For  $l+1 < k < 2l$ ,  $f_{NL}$  decreases and reaches  $f_{NL} = 1$  at  $k = 2l$ . The non-Gaussianity parameter  $f_{NL}$  takes a large value transiently. So if we want to obtain the large  $f_{NL}$  from the present observation, we need  $\rho_2$  to decay into radiation at  $k = l+1$ . Next we consider the period when the second component  $\rho_2$  gets to dominate the cosmic energy density  $\rho$ . In this period, the first component  $\rho_1$  is subdominant. So we can use (4.40) and get

$$\begin{aligned} N = & N(1) + \frac{1}{3} \ln \left[ \frac{m_2^2 \phi_2^2(0)}{2\rho} \left\{ \frac{\kappa^2 \rho_1(1)}{3m_2^2} \right\}^{3/4} \frac{1}{(1 - \kappa^2 \phi_2^2(0)/6)^{3/4}} \right] \\ & + \frac{1}{3} \left( \frac{2}{m_2^2 \phi_2^2(0)} \right)^{4/3} \frac{3m_2^2}{\kappa^2} \left( 1 - \frac{\kappa^2 \phi_2^2(0)}{6} \right) \rho^{1/3}, \end{aligned} \quad (4.48)$$

which is simplified into the model (4.28) for small  $\rho$ . Then we can obtain the evaluation of  $f_{NL}$  (4.30)(4.31)(4.32). If we want  $f_{NL}$  of order of unity, we need  $l > 1$ .

There is a case where the contribution to  $N$  from the second component  $\rho_2(1)$  dependent term increases but is bounded in spite of  $g_2 < g_1$ . In this case, the non-Gaussianity parameter  $f_{NL}$  cannot grow into a significant value. We consider the case where  $\phi_a$  ( $a = 1, 2$ ) with mass  $m_a$  ( $m_2^2 < m_1^2$ ) are in the slow rolling phase. In this case,  $g_a = 2m_a^2/\kappa^2\rho$ . Unlike the previous case,  $g_a$  depends on the cosmic energy density  $\rho$ . The ratio of the energy density of the second component  $s_2$  and the logarithm of the scale factor  $N$  are given by

$$s_2 = \frac{\rho_2(0)}{\rho(0)} \left( \frac{\rho}{\rho(0)} \right)^{-1+m_2^2/m_1^2}, \quad (4.49)$$

$$N = -\frac{\kappa^2}{2m_1^2}(\rho - \rho(0)) + \frac{\kappa^2}{2m_1^2} \frac{-m_1^2 + m_2^2}{m_2^2} \rho_2(0) \left[ \left( \frac{\rho}{\rho(0)} \right)^{m_2^2/m_1^2} - 1 \right] \quad (4.50)$$

where 0 in  $\rho(0)$ ,  $\rho_a(0)$  implies the first horizon crossing time and  $\rho(0) = \rho_1(0) + \rho_2(0)$  where  $\rho_a(0) = m_a^2 \phi_a^2(0)/2$ . From the above expression of  $N$ , we can verify that the  $\rho_2(0)$  dependent term is bounded and suppressed by  $\kappa^2 \phi_2^2(0)$ , therefore  $f_{NL}$  is suppressed by  $10^{-2}$ .

#### 4.2.2 the case with the cosmological term

In cases where the cosmological term  $U_0$  exists, we use  $\sigma := \rho - U_0$  as the evolution parameter. In these cases, the evolution equations corresponding to (3.11), (3.12) are the same evolution equations (3.11), (3.12) but all  $\rho$ 's are replaced with  $\sigma$ 's.  $s_A$  is defined by  $s_A := \rho_A/\sigma$  and satisfies  $\sum_A s_A = 1$ . When  $|s_2| \ll 1$ , linearizing with respect to  $s_2$  gives

(4.33), (4.34) but all  $\rho$ 's are replaced with  $\sigma$ 's. In the same way, when  $|s_1| \ll 1$ , linearizing with respect to  $s_1$  gives (4.37), (4.38) but all  $\rho$ 's are replaced with  $\sigma$ 's.

We consider two scalar fields  $\phi_a$  ( $a = 1, 2$ ) which move on the potential given by

$$\rho = U_0 + \sum_{a=1}^2 \rho_a, \quad \rho_a = \frac{1}{2} \eta_a \phi_a^2. \quad (4.51)$$

In this case, the  $g$  factor of the scalar field  $\phi_a$  ( $a = 1, 2$ ) is given by  $g_a = 2\eta_a/\kappa^2 \rho \cong 2\eta_a/\kappa^2 U_0$ . In order that the inflation can solve the horizon problem, we assume that  $g_a \sim 10^{-2}$ . We assume that  $\eta_1 > \eta_2$ . First the energy of the scalar field  $\phi_1$ ,  $\rho_1$  dominate  $\sigma$  and next the energy of the scalar field  $\phi_2$ ,  $\rho_2$  grows and gets to dominate  $\sigma$ . In the first period where  $\rho_1$  dominates  $\sigma$ ,  $N$  is given by

$$N = -\frac{\kappa^2 U_0}{2\eta_1} \ln \frac{\sigma}{\rho_1(0)} + \frac{\kappa^2 U_0}{2\eta_1} \frac{\rho_2(0)}{\rho_1(0)} \left( \frac{\rho_1(0)}{\sigma} \right)^{1-\eta_2/\eta_1}, \quad (4.52)$$

where

$$\rho_a(0) := \frac{1}{2} \eta_a \phi_a^2(0). \quad (4.53)$$

In the second period where  $\rho_2$  dominates  $\sigma$ ,  $N$  is given by

$$N = -\frac{\kappa^2 U_0}{2\eta_2} \ln \frac{\sigma}{\rho_2(0)} + \frac{\kappa^2 U_0}{2\eta_2} \frac{\rho_1(0)}{\rho_2(0)} \left( \frac{\rho_2(0)}{\sigma} \right)^{1-\eta_1/\eta_2}, \quad (4.54)$$

which can be obtained when all the subscripts 1, 2 are exchanged in the expression of  $N$  in the first period (4.52).

We consider the case that  $\eta_1 > 0$ ,  $\eta_2 < 0$ . In this case, in the first period  $\sigma$  is positive, decreases for the time evolution, and then the  $\rho_2(0)$  dependent term in  $N$  (4.52) grows boundlessly. In the first period,  $N$  can be simplified into the model given by

$$N = 10^2 \ln \phi_1(0) - \kappa^2 \alpha(\sigma) \phi_2^2(0), \quad (4.55)$$

up to the  $\sigma$  dependent part which is not related with the Bardeen parameters  $\zeta_n(\rho)$ , which yields the non-Gaussianity parameter  $f_{NL}$  given by

$$f_{NL} = \left( -10^6 \frac{1}{\phi_1^4(0)} - \kappa^6 \alpha^3(\sigma) \phi_2^2(0) \right) / \left( 10^4 \frac{1}{\phi_1^2(0)} + \kappa^4 \alpha^2(\sigma) \phi_2^2(0) \right)^2. \quad (4.56)$$

We put

$$\phi_1(0) = \frac{1}{\kappa} 10^m, \quad \phi_2(0) = \frac{1}{\kappa} 10^{-l}, \quad \alpha(\sigma) = 10^k, \quad (4.57)$$

where  $k$  grows for the time evolution and  $k < 2l+2$  for the condition that  $\rho_2$  is subdominant. The non-Gaussianity parameter  $f_{NL}$  is written by

$$f_{NL} = \frac{-10^{6-4m} - 10^{3k-2l}}{(10^{4-2m} + 10^{2k-2l})^2}. \quad (4.58)$$



The exponent evaluation method gives

$$k < k_1 \quad f_{NL} = -10^{-2}, \quad (4.59)$$

$$k_1 < k < k_2 \quad f_{NL} = -10^{3k-2l-8+4m}, \quad (4.60)$$

$$k_2 < k < k_3 \quad f_{NL} = -10^{-k+2l}, \quad (4.61)$$

where

$$k_1 := 2 - \frac{4}{3}m + \frac{2}{3}l, \quad k_2 := 2 - m + l, \quad k_3 := 2l + 2. \quad (4.62)$$

For  $k_1 < k < k_2$ , the absolute value of  $f_{NL}$  grows and reaches the maximum  $f_{NL} = 10^{-2+m+l}$  at  $k = k_2$ . For  $k_2 < k < k_3$ , the absolute value of  $f_{NL}$  decreases and reaches  $f_{NL} = 10^{-2}$  at  $k = k_3$ . In the second period,  $\sigma$  is negative,  $-\sigma$  grows, and the  $\rho_1(0)$  dependent term in  $N$  (4.54) decays.  $N$  in the second period (4.54) can be simplified into the model given by

$$N = -10^2 \ln \phi_2(0) + \kappa^2 \beta(\sigma) \phi_1^2(0), \quad (4.63)$$

up to the  $\sigma$  dependent part which is not related with the Bardeen parameters  $\zeta_n(\rho)$ . Putting

$$\phi_1(0) = \frac{1}{\kappa} 10^m, \quad \phi_2(0) = \frac{1}{\kappa} 10^{-l}, \quad \beta(\sigma) = 10^{-k}, \quad (4.64)$$

where  $k$  increases for the time evolution and satisfies  $k > 2m - 2$  for the condition that  $\rho_1$  is subdominant, we obtain

$$\begin{aligned} f_{NL} &= \left( 10^6 \frac{1}{\phi_2^4(0)} + \kappa^6 \beta^3(\sigma) \phi_1^2(0) \right) / \left( 10^4 \frac{1}{\phi_2^2(0)} + \kappa^4 \beta^2(\sigma) \phi_1^2(0) \right)^2 \\ &= \frac{10^{6+4l} + 10^{-3k+2m}}{(10^{2l+4} + 10^{-2k+2m})^2} \\ &\cong 10^{-2}. \end{aligned} \quad (4.65)$$

We consider the case  $\eta_1, \eta_2 < 0$  ( $\eta_1 > \eta_2$ ). In the first period,  $\sigma$  is negative,  $-\sigma$  increases, and the  $\rho_2(0)$  dependent term in  $N$  (4.52) grows boundlessly. In the first period,  $N$  (4.52) can be simplified into the model given by

$$N = -10^2 \ln \phi_1(0) - \kappa^2 \alpha(\sigma) \phi_2^2(0), \quad (4.66)$$

up to the  $\sigma$  dependent part which is not related with the Bardeen parameters  $\zeta_n(\rho)$ , which yields the non-Gaussianity parameter  $f_{NL}$  given by

$$\begin{aligned} f_{NL} &= \left( 10^6 \frac{1}{\phi_1^4(0)} - \kappa^6 \alpha^3(\sigma) \phi_2^2(0) \right) / \left( 10^4 \frac{1}{\phi_1^2(0)} + \kappa^4 \alpha^2(\sigma) \phi_2^2(0) \right)^2 \\ &= \frac{10^{6+4m} - 10^{3k-2l}}{(10^{2m+4} + 10^{2k-2l})^2}, \end{aligned} \quad (4.67)$$

by putting

$$\phi_1(0) = \frac{1}{\kappa} 10^{-m}, \quad \phi_2(0) = \frac{1}{\kappa} 10^{-l}, \quad \alpha(\sigma) = 10^k, \quad (4.68)$$

where in the first period  $\rho_1$  is dominant  $m < l$  and  $k$  increases for the time evolution,  $k < 2l + 2$  from the condition that  $\rho_2$  is subdominant. The exponent evaluation method gives

$$k < k_1 \quad f_{NL} = 10^{-2}, \quad (4.69)$$

$$k_1 < k < k_2 \quad f_{NL} = -10^{3k-2l-4m-8}, \quad (4.70)$$

$$k_2 < k < k_3 \quad f_{NL} = -10^{-k+2l}, \quad (4.71)$$

where

$$k_1 := \frac{2}{3}l + \frac{4}{3}m + 2, \quad k_2 := l + m + 2, \quad k_3 := 2l + 2. \quad (4.72)$$

For  $k_1 < k < k_2$ , the absolute value of  $f_{NL}$  grows, reaches the maximum  $f_{NL} = -10^{l-m-2}$  at  $k = k_2$ . For  $k_2 < k < k_3$ , the absolute value of  $f_{NL}$  decreases, reaches  $f_{NL} = -10^{-2}$  at  $k = k_3$ . In the second period,  $\sigma$  is negative,  $-\sigma$  grows and the  $\rho_1(0)$  dependent term in  $N$  (4.54) decays.  $N$  in the second period (4.54) can be simplified into the model given by

$$N = -10^2 \ln \phi_2(0) - \kappa^2 \beta(\sigma) \phi_1^2(0), \quad (4.73)$$

up to the  $\sigma$  dependent part which is not related with the Bardeen parameters  $\zeta_n(\rho)$ , which yields

$$\begin{aligned} f_{NL} &= \left( 10^6 \frac{1}{\phi_2^4(0)} - \kappa^6 \beta^3(\sigma) \phi_1^2(0) \right) / \left( 10^4 \frac{1}{\phi_2^2(0)} + \kappa^4 \beta^2(\sigma) \phi_1^2(0) \right)^2 \\ &= \frac{10^{6+4l} - 10^{-3k-2m}}{(10^{2l+4} + 10^{-2k-2m})^2} \\ &\cong 10^{-2}, \end{aligned} \quad (4.74)$$

putting

$$\phi_1(0) = \frac{1}{\kappa} 10^{-m}, \quad \phi_2(0) = \frac{1}{\kappa} 10^{-l}, \quad \beta(\sigma) = 10^{-k}, \quad (4.75)$$

where  $k$  grows for the time evolution, and  $k > -2m - 2$  from the condition that  $\rho_1$  is subdominant. The above results about two scalar fields in the vacuum domination can be investigated by the method of the  $\tau$  function. [9] The same results given in this subsection are reproduced using the  $\tau$  function in the Appendix C.

The mechanism which produces the large  $f_{NL}$  depends on the fact that  $g_1 > g_2$  and that  $s_1$  is dominant in the first time and that  $s_2$  begins to govern the cosmic energy density  $\rho$  gradually. So when  $s_1$  is the radiation  $g_1 = 4$ , and  $s_2$  is the scalar field with the negative mass  $g_2 = 2\eta/\kappa^2 U_0$  ( $\eta < 0$ ), the non-Gaussianity  $f_{NL}$  can grow transiently, because the third term which depends on  $\rho_2(0)$

$$N = N(0) - \frac{1}{4} \ln \frac{\sigma}{\rho_1(0)} + \frac{1}{4} \frac{\rho_2(0)}{\rho_1(0)} \left( \frac{\rho_1(0)}{\sigma} \right)^{1-\eta/2\kappa^2 U_0}, \quad (4.76)$$

can grow boundlessly.

## §5 Discussion

In this paper, the first half, that is sections 2, 3 is devoted to the general considerations about the gauge invariant nonlinear cosmological perturbation theory on superhorizon scales and the latter half, that is section 4 is devoted to the investigations of the concrete physical systems. Here we discuss how the general theory in the first half is used in the analysis of the concrete physical systems in the latter half.

(A) In subsection 2.1, we gave the definitions of all the types of gauge invariant perturbation variables. The numerical data obtained by the cosmological observations are related with the gauge invariant perturbation variables, since both quantities do not depend on how we set up the spacetime coordinate system. Therefore it is desirable to express all the physical laws in the form closed by the gauge invariant quantities only. By the subsection 2.1, it is guaranteed that the perturbation variables given by operating  $D(\rho)$  derivatives on the scalar like objects such as  $\ln a$ ,  $s_A := \rho_A/\rho$  are gauge invariant. In the analysis of the concrete physical systems in the latter half of the present paper, the adiabatic perturbation variable  $D(\rho)^n \ln a$ , the entropic perturbation variables  $D(\rho)^n s_A$  are used as the independent variables and the total energy density  $\rho$  is used as the evolution parameter. This formalism by the A/E decomposition given in the subsection 3.2 and the  $\rho$  philosophy proposed in the subsection 3.3 is very useful for the physical interpretations of the results obtained in the latter half of the paper.

(B) In the subsection 2.2, we construct the metric junction formalism as the method treating the sudden change of the equation of state. The metric junction formalism given in the subsection 2.2 is used in the analysis of the concrete physical systems in the latter half of the paper, since they contain the transitions such as the slow rolling oscillatory transition and the reheating transition. Since it is proven in the subsection 2.2 that our A/E perturbation variables  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$  are continuous across the matching surface defined by  $\rho = \text{const}$ , this set of variables must be useful in the research of the transitions even if we assume that the transition do not occur instantly.

(C) Based on the  $\rho$  philosophy proposed in the subsection 3.3, in section 4 the evolutions of  $\ln a$ ,  $s_A$  are described as the functions of the evolution parameter  $\rho$ . The evolutions of our A/E perturbation variables  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$  are given by operating  $D(\rho)$  derivatives on the solutions of  $\ln a$ ,  $s_A$  expressed as the functions of  $\rho$ . Since  $D(\rho)f(\rho) = 0$  for an arbitrary function of  $\rho$ ;  $f(\rho)$ , the  $\rho$  dependences of  $\ln a$ ,  $s_A$  are directly reflected to the  $\rho$  dependences of  $D(\rho)^n \ln a$ ,  $D(\rho)^n s_A$ . For this reason, by the formalism prepared in section 3, in section 4 we can manifestly clarify the time evolutions in which the entropic perturbation of the energetically subdominant component makes the adiabatic perturbation  $D(\rho)^n \ln a$  grow. We can show that  $D(\rho)^n \ln a$ , the non-Gaussianity  $f_{NL}$  grow considerably and transiently when the energy ratio  $s_2$  of the energetically subdominant component which has the  $g$  factor smaller than the  $g$  factor of the dominant component  $\rho_1$  and which is supported by the extremely small scalar field expectation value, begins to increase. Since until now as for  $D(\rho)^n \ln a$  only expression of the final state when the growth of  $D(\rho)^n \ln a$  has already ended, has been given, the  $\rho$  philosophy given in section 3 is superior in that the time evolution and the mechanism of the growth of the adiabatic perturbation variable called the Bardeen parameters  $D(\rho)^n \ln a$ , the non-Gaussianity  $f_{NL}$  can be described.

In the following paragraphs, the two points which was not treated in the first section are discussed. The first point is on the existence of more than two sources of cosmological

perturbations. The second point is on the modeling of the abrupt cosmic evolution by the metric junction across the spacelike hypersurface.

(1) The present amplitude of the Bardeen parameter  $\zeta_n(\rho)$  can be decomposed as  $\zeta_n(\rho) = \zeta_{n\text{sl}}(\rho) + \zeta_{n\text{ent}}(\rho)$ .  $\zeta_{n\text{sl}}(\rho)$  is the component generated from the adiabatic mode in the slow rolling phase, and  $\zeta_{n\text{ent}}(\rho)$  is the adiabatic component generated from the entropy mode in the slow rolling phase by successive universe evolution. In many excellent papers, many authors insisted that a significant large nonlinearity can be generated in the inhomogeneous end of the inflation [28] [29], in the modulated reheating [31], in the curvaton scenario [32] and in the vacuum dominated inflation [33] [34]. Unfortunately in the partial studies, without any plausible reasons it is assumed that  $\zeta_{n\text{sl}}(\rho)$  is negligibly small compared with  $\zeta_{n\text{ent}}(\rho)$  and the non-Gaussianity parameter  $f_{NL}$  is calculated from  $\zeta_{n\text{ent}}(\rho)$  only. However the inflaton which drives the universe expansion enough to solve the horizon problem, the flatness problem, but do not generate any cosmological perturbations, does not exist. In this point of view, it is wonderful that the authors of the paper [32] tried to treat the contribution of the inflaton  $\zeta_{n\text{sl}}(\rho)$  and the contribution of the curvaton  $\zeta_{n\text{ent}}(\rho)$  with equal importance from the standpoint of the mixed scenario. In this paper, we investigate whether a significant non-Gaussianity  $f_{NL}$  is generated in the successive universe evolution taking into account the cosmological perturbations generated in the slow rolling phase. When we analyze the systems where more than two factors are concerned, the exponent evaluation method presented in this paper is very efficient.

(2) In the early universe, there exists a period before which and after which the dynamical behaviors of each component are very different. As for the scalar field with mass  $m$ , while  $m \ll H$ , where  $H$  is the Hubble parameter, holds, the scalar field is in the slow rolling phase when its energy density changes mildly compared with the cosmic expansion, and while  $H \ll m$  holds, the scalar field is in the oscillatory phase when its energy density behaves like a dust fluid.[3] [6] [7] [8] In the reheating [4] [8], the energy of the oscillatory scalar field behaving like a dust fluid is transformed into that of the radiation fluid. In the hybrid inflation, the energy of the slow rolling scalar field is transformed into that of the oscillatory scalar field and into that of the radiation fluid on the bifurcation set. Such phase transitions are quite complicated and the completely rigorous mathematical treatment is beyond our scope. For example, we consider the slow rolling-oscillatory transition. In the  $m \ll H$  region and in the  $H \ll m$  region, the expansion schemes investigating the dynamical behaviors of the scalar field can be developed with  $m/H$  and  $H/m$  as the expansion parameter, respectively.[9] [3] [6] [7] [8] But at the transition period  $H \sim m$ , any expansion schemes cannot be developed because of no expansion parameters. However, in spite of complicated behaviors at the transition, the period of the transition can be thought to be short compared with the periods before and after the transition characterized by  $m \ll H$ ,  $H \ll m$ , respectively. Therefore we think the transitions as the instantaneous transient phenomena and may treat such transitions as the metric junctions across the spacelike hypersurfaces. In the above reasons, in our paper, the metric junction formulation on the cosmological perturbations in the long wavelength limit, linear and nonlinear, are constructed. On the spacelike hypersurface defined by  $H = m$ , the spacetime governed by the slow rolling scalar field and the spacetime governed by the oscillatory scalar field is connected. In case of reheating, On the spacelike hypersurface defined by  $H = \Gamma$  where  $\Gamma$  is the decay constant of the scalar field, the spacetime governed by the oscillatory scalar

field and the spacetime governed by the radiation fluid is connected.

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## §A Proof of Proposition 8

Solving the matching condition  $C(t, \mathbf{x}, \lambda) = 0$  with respect to  $t$  gives  $t = t(\mathbf{x}, \lambda)$ . By differentiating  $C(t, \mathbf{x}, \lambda) = 0$  with respect to  $\lambda, x^k$ , we obtain

$$\frac{dt}{d\lambda} = -\frac{1}{\dot{C}} \frac{dC}{d\lambda}, \quad \frac{\partial t}{\partial x^k} = -\frac{1}{\dot{C}} \partial_k C. \quad (\text{A.1})$$

Differentiating  $[S]_{\pm}^{\pm} = 0$  with respect to  $\lambda, x^k$  gives

$$\left[ \frac{dS}{d\lambda} + \dot{S} \frac{dt}{d\lambda} \right]_{\pm}^{+} = 0, \quad \left[ \partial_i S + \dot{S} \frac{\partial t}{\partial x^i} \right]_{\pm}^{+} = 0 \quad (\text{A.2})$$

From the two sets of equations, we obtain  $[D(C)S]_{\pm}^{\pm} = 0, [D_i(C)S]_{\pm}^{\pm} = 0$ .

## §B the junction condition in the linear perturbation theory

In this section, we consider the metric junction across the matching hypersurface characterized by  $\tilde{C} = 0$  where  $\tilde{C}$  is the scalar quantity in the linear perturbation theory, since the previous papers treating this problem [24 ] [25 ] [26 ] sometimes contain some typographical errors, derive the matching conditions without keeping the gauge invariance completely and are lacking in the physical interpretations from the viewpoint of the long wavelength limit. The notations used in this section are based on the papers. [17 ] [5 ] [8 ] The metric tensor is given by

$$\tilde{g}_{00} = -(1 + 2AY), \quad (\text{B.1})$$

$$\tilde{g}_{0i} = -aBY_i, \quad (\text{B.2})$$

$$\tilde{g}_{ij} = a^2[(1 + 2H_L Y)\delta_{ij} + 2H_T Y_{ij}], \quad (\text{B.3})$$

where  $Y, Y_i$  and  $Y_{ij}$  are harmonic scalar, vector and tensor for a scalar perturbation with wavenumber  $\mathbf{k}$ :

$$Y := e^{i\mathbf{k}\mathbf{x}}, \quad Y_i := -i\frac{k_i}{k}Y, \quad Y_{ij} := \left( \frac{1}{3}\delta_{ij} - \frac{k_i k_j}{k^2} \right) Y, \quad (\text{B.4})$$

where  $k^2 := \sum_i k_i k_i$ . The energy momentum tensor of the total system is given by

$$\tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{P})\tilde{u}_{\mu}\tilde{u}_{\nu} + \tilde{P}\tilde{g}_{\mu\nu}, \quad (\text{B.5})$$

where  $\tilde{\rho}$ ,  $\tilde{P}$  and  $\tilde{u}_\mu$  are the energy density, the pressure, and the four velocity of the total system. For the scalar quantities  $\tilde{S} = (\tilde{\rho}, \tilde{P})$ ,  $\tilde{S}$  is expanded as  $\tilde{S} = S + \delta SY$ , and the four velocity  $\tilde{u}_\mu$  is written by

$$\tilde{u}_0 = -(1 + AY), \quad (\text{B.6})$$

$$\tilde{u}_i = a(v - B)Y_i. \quad (\text{B.7})$$

We define the gauge dependent geometrical quantities as

$$\mathcal{R} := H_L + \frac{1}{3}H_T, \quad \sigma_g := \frac{a}{k}\dot{H}_T - B. \quad (\text{B.8})$$

For the scalar quantity  $\tilde{S}$ ,

$$DS := \delta S - \frac{\dot{S}}{H}\mathcal{R} \quad (\text{B.9})$$

is gauge invariant. For the four velocity  $\tilde{u}_\mu$ , the variable defined by

$$Z := \mathcal{R} - \frac{aH}{k}(v - B) \quad (\text{B.10})$$

is gauge invariant. The Newtonian potential  $\Phi$  defined by

$$\Phi := \mathcal{R} - \frac{aH}{k}\sigma_g \quad (\text{B.11})$$

is gauge invariant.

We consider the metric junction across the hypersurface  $\Sigma$  defined by

$$x^0 = t_\times + \delta Z^0 Y, \quad x^i = y^i + \delta Z Y^i, \quad (\text{B.12})$$

which connects the future spacetime  $\mathcal{M}_+$  and the past spacetime  $\mathcal{M}_-$ . From  $\delta Z^0$ ,  $\delta Z$ , we can define two gauge invariant quantities as

$$\phi^0 := \delta Z^0 + \frac{1}{H}\mathcal{R}, \quad (\text{B.13})$$

$$\phi := \delta Z - \frac{1}{k}H_T. \quad (\text{B.14})$$

We consider the case where the matching hypersurface  $\Sigma$  is defined by  $\tilde{C} = 0$ . In this case

$$\phi^0 = -\frac{1}{\tilde{C}}DC. \quad (\text{B.15})$$

The normal vector of the matching hypersurface which points from  $\mathcal{M}_-$  to  $\mathcal{M}_+$  is given by

$$\tilde{n}_\mu = -\text{sgn}(\dot{\tilde{T}})[- \tilde{g}^{\rho\sigma} \partial_\rho \tilde{T} \partial_\sigma \tilde{T}]^{-1/2} \partial_\mu \tilde{T}. \quad (\text{B.16})$$

We define the intrinsic metric, the extrinsic curvature, the intrinsic energy momentum tensor by

$$\tilde{q}_{ij} := \tilde{e}_i^\mu \tilde{e}_j^\nu (\tilde{g}_{\mu\nu} + \tilde{n}_\mu \tilde{n}_\nu), \quad (\text{B.17})$$

$$\tilde{K}_{ij} := \tilde{e}_i^\mu \tilde{e}_j^\nu \tilde{\nabla}_\mu \tilde{n}_\nu, \quad (\text{B.18})$$

$$\tilde{T}_{nn} := \tilde{n}^\mu \tilde{n}^\nu \tilde{T}_{\mu\nu}, \quad (\text{B.19})$$

$$\tilde{T}_{ni} := \tilde{n}^\mu \tilde{e}_i^\nu \tilde{T}_{\mu\nu}, \quad (\text{B.20})$$

where  $\tilde{e}_i^\mu := \partial x^\mu / \partial y^i$ . These quantities of the matching hypersurface  $\Sigma$  defined by  $\tilde{C} = 0$  can be written as

$$\tilde{q}_{ij} = a^2 \left\{ \delta_{ij} + 2\delta_{ij}Y \left( -H \frac{DC}{\dot{C}} + \frac{k}{3}\phi \right) + 2Y_{ij}(-k\phi) \right\}, \quad (\text{B.21})$$

$$\begin{aligned} \tilde{K}_{ij} &= a^2 H \delta_{ij} \\ &+ \left\{ \left( -2a^2 H^2 - a^2 \dot{H} + \frac{k^2}{3} \right) \frac{DC}{\dot{C}} + \frac{2}{3} a^2 H k \phi + \frac{a^2}{2} H \frac{D\rho}{\rho} \right\} Y \delta_{ij} \\ &+ \left( -2a^2 H k \phi - k^2 \frac{DC}{\dot{C}} - \frac{k^2}{H} \Phi \right) Y_{ij}, \end{aligned} \quad (\text{B.22})$$

$$\tilde{T}_{nn} = \rho + \left( -\frac{\dot{\rho}}{\dot{C}} DC + D\rho \right) Y, \quad (\text{B.23})$$

$$\tilde{T}_{ni} = (\rho + P) \left( k \frac{DC}{\dot{C}} + \frac{k}{H} Z \right), \quad (\text{B.24})$$

where right hand sides are written in the gauge invariant form. The metric junction conditions across the matching hypersurface  $\Sigma$  are given by

$$[\tilde{q}_{ij}]_-^+ = [\tilde{K}_{ij}]_-^+ = [\tilde{T}_{nn}]_-^+ = [\tilde{T}_{ni}]_-^+ = 0, \quad (\text{B.25})$$

which yield the matching conditions in the long wavelength limit: in the background level,

$$[a]_-^+ = [H]_-^+ = [\rho]_-^+ = 0, \quad (\text{B.26})$$

and in the perturbation level,

$$[\phi]_-^+ = 0, \quad (\text{B.27})$$

$$[k^2 \Phi]_-^+ = 0, \quad (\text{B.28})$$

$$\left[ \frac{DC}{\dot{C}} \right]_-^+ = \left[ -\frac{\dot{\rho}}{\dot{C}} DC + D\rho \right]_-^+ = 0, \quad (\text{B.29})$$

$$\left[ (\rho + P) \left( \frac{DC}{\dot{C}} + \frac{1}{H} Z \right) \right]_-^+ = 0. \quad (\text{B.30})$$

Owing to our previous paper [8], in the long wavelength limit, the solution of  $DS$  where  $S$  is the scalar quantity is given by

$$DS = DS^\sharp + \frac{\dot{S}}{H} c \int_{t_0} dt \frac{1}{a^3}, \quad (\text{B.31})$$

where

$$DS^\sharp := \frac{\partial S}{\partial C_\star} - \frac{\dot{S}}{\dot{a}} \frac{\partial a}{\partial C_\star} \quad (\text{B.32})$$

where  $\partial S/\partial C_\star$ ,  $\partial a/\partial C_\star$  are the derivatives of the background quantities  $S$ ,  $a$  with respect to the solution constant  $C_\star$  and  $c$  is a constant characterizing the adiabatic decaying mode. The solution of the Newtonian potential  $\Phi$  is given by

$$k^2\Phi = \frac{3H}{a}c + O(k^2). \quad (\text{B.33})$$

Therefore (B.28), (B.29) give

$$[c]_-^+ = 0, \quad (\text{B.34})$$

$$[D(C)a]_-^+ = [D(C)\rho]_-^+ = 0, \quad (\text{B.35})$$

where as for scalar like object  $S$

$$D(C)S := \frac{\partial S}{\partial C_\star} - \frac{\dot{S}}{\dot{C}} \frac{\partial C}{\partial C_\star}. \quad (\text{B.36})$$

(B.34), (B.35) are consistent with the metric junction conditions of the full nonlinear gradient expansion case represented by (2.66) and the Proposition 8.

## §C the analyses of the evolution of the multiple vacuum dominated scalar fields by the $\tau$ function

The  $\tau$  function was presented as the method of analyzing the evolution of the multiple scalar fields. [9] Under the slow rolling approximation, the evolution of the scalar fields is described by

$$\frac{d\phi_a}{dN} = -\frac{1}{\kappa^2 U} \frac{\partial U}{\partial \phi_a}, \quad (\text{C.1})$$

which are decomposed as

$$\frac{d\phi_a}{d\tau} = -\frac{\partial U}{\partial \phi_a}, \quad \frac{dN}{d\tau} = \kappa^2 U, \quad (\text{C.2})$$

introducing the  $\tau$  function as the new evolution parameter.[9] It is much easier to treat the new evolution equations (C.2) than to treat the original evolution equations (C.1) for many cases.

We consider the vacuum dominated case given by

$$U = U_0 + \sum_a \frac{1}{2} \eta_a \phi_a^2. \quad (\text{C.3})$$

In this case, the evolution of  $\phi_a$  is given by

$$\phi_a = \phi_a(0) \exp(-\eta_a \tau). \quad (\text{C.4})$$

By using the  $\tau$  function as the evolution parameter, the Bardeen parameter  $\zeta_n(\rho)$  are given by

$$\zeta_n(\rho) = \left( \frac{\partial}{\partial \lambda} - \frac{U_\lambda}{U_\tau} \frac{\partial}{\partial \tau} \right)^n \left( \kappa^2 \int_0^\tau d\tau U \right), \quad (\text{C.5})$$



where the subscripts  $\lambda, \tau$  are interpreted as the derivatives with respect to  $\lambda, \tau$ , respectively; for example

$$U_{\lambda\tau} := \frac{\partial}{\partial\lambda} \frac{\partial}{\partial\tau} U. \quad (\text{C.6})$$

Concretely  $\zeta_1(\rho), \zeta_2(\rho)$  are given by

$$\frac{1}{\kappa^2} \zeta_1(\rho) = \int_0^1 d\tau U_\lambda - \frac{U_\lambda}{U_\tau} U, \quad (\text{C.7})$$

$$\frac{1}{\kappa^2} \zeta_2(\rho) = \int_0^1 d\tau U_{\lambda\lambda} - \frac{U_\lambda^2}{U_\tau} + \frac{U}{U_\tau^2} \left( -U_{\lambda\lambda} U_\tau + 2U_\lambda U_{\lambda\tau} - \frac{U_\lambda^2}{U_\tau} U_{\tau\tau} \right), \quad (\text{C.8})$$

and the coefficients  $N_a, N_{ab}$  are given by the expansions

$$\zeta_1(\rho) = \sum_a N_a \frac{d\phi_a(0)}{d\lambda}, \quad \zeta_2(\rho) = \sum_{ab} N_{ab} \frac{d\phi_a(0)}{d\lambda} \frac{d\phi_b(0)}{d\lambda}, \quad (\text{C.9})$$

where we assume that all the nonlinear perturbations at the first horizon crossing in the inflationary expansion are vanishing:  $d^n \phi_a(0)/d\lambda^n = 0$  ( $n \geq 2$ ). By using  $A(n, k)$  defined by

$$A(n, k) := \sum_a \eta_a^n \phi_a^2(0) \exp(-2k\eta_a\tau), \quad (\text{C.10})$$

and collecting the leading order terms with respect to  $U_0$ , we obtain

$$f_{NL} = \frac{1}{\kappa^2 U_0} \left[ \frac{A(2, 1)A(3, 3)}{A(2, 2)^2} - 4 \frac{A(3, 2)}{A(2, 2)} + 2 \frac{A(3, 1)}{A(2, 1)} \right]. \quad (\text{C.11})$$

We consider the system where two scalar fields evolve. Since only the first term in (C.11) can change the exponent for the moving  $\tau$ , we concentrate on the first term written by  $(f_{NL})_1$  from now on.  $(f_{NL})_1$  can be written by

$$(f_{NL})_1 = \frac{\eta_1}{\kappa^2 U_0} \frac{(1 + \eta_r^2 \phi_r^2 e_r)(1 + \eta_r^3 \phi_r^2 e_r^3)}{(1 + \eta_r^2 \phi_r^2 e_r^2)^2}, \quad (\text{C.12})$$

where

$$\eta_r := \frac{\eta_2}{\eta_1}, \quad \phi_r := \frac{\phi_2(0)}{\phi_1(0)}, \quad e_r := \exp\{-2(\eta_2 - \eta_1)\tau\}. \quad (\text{C.13})$$

In order that the inflation can solve the horizon problem, we assume  $\eta_a/\kappa^2 U_0 \sim 10^{-2}$ .

First we consider the case  $\eta_1 > 0, \eta_2 < 0$ . we put

$$\phi_1(0) = \frac{1}{\kappa} 10^m, \quad \phi_2(0) = \frac{1}{\kappa} 10^{-l}, \quad e_r = 10^p, \quad (\text{C.14})$$

then we get

$$(f_{NL})_1 = 10^{-2} \frac{(1 + 10^{-2(l+m)+p})(1 - 10^{-2(l+m)+3p})}{(1 + 10^{-2(l+m)+2p})^2}. \quad (\text{C.15})$$

The exponent evaluation method gives

$$p < p_1 \quad (f_{NL})_1 = 10^{-2}, \quad (\text{C.16})$$

$$p_1 < p < p_2 \quad (f_{NL})_1 = -10^{3p-2(l+m)-2}, \quad (\text{C.17})$$

$$p_2 < p < p_3 \quad (f_{NL})_1 = -10^{-p+2(l+m)-2}, \quad (\text{C.18})$$

$$p_3 < p \quad (f_{NL})_1 = -10^{-2}, \quad (\text{C.19})$$

where

$$p_1 := \frac{2}{3}(l+m), \quad p_2 := (l+m), \quad p_3 := 2(l+m). \quad (\text{C.20})$$

The above evaluations agree completely with (4.59), (4.60), (4.61) by taking the correspondence  $p = k + 2m - 2$ .

Next we consider the case where  $\eta_a < 0$ ,  $\eta_1 > \eta_2$ . We put

$$\phi_1(0) = \frac{1}{\kappa}10^{-m}, \quad \phi_2(0) = \frac{1}{\kappa}10^{-l}, \quad e_r = 10^p, \quad (\text{C.21})$$

where  $m < l$ , then we obtain

$$(f_{NL})_1 = -10^{-2} \frac{(1 + 10^{-2(l-m)+p})(1 + 10^{-2(l-m)+3p})}{(1 + 10^{-2(l-m)+2p})^2}. \quad (\text{C.22})$$

The exponent evaluation method gives

$$p < p_1 \quad (f_{NL})_1 = -10^{-2}, \quad (\text{C.23})$$

$$p_1 < p < p_2 \quad (f_{NL})_1 = -10^{3p-2(l-m)-2}, \quad (\text{C.24})$$

$$p_2 < p < p_3 \quad (f_{NL})_1 = -10^{-p+2(l-m)-2}, \quad (\text{C.25})$$

$$p_3 < p \quad (f_{NL})_1 = -10^{-2}, \quad (\text{C.26})$$

where

$$p_1 := \frac{2}{3}(l-m), \quad p_2 := (l-m), \quad p_3 := 2(l-m). \quad (\text{C.27})$$

The above evaluations agree completely with (4.69), (4.70), (4.71) by taking the correspondence  $p = k - 2m - 2$ .

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